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Travelling-wave solutions for parabolic systems

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Summary

This thesis is concerned with travelling-wave solutions to parabolic systems. It is known that the system of N reaction-diffusion equations,

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(u),$$

has monotone travelling-wave solutions connecting two equilibria, S and T say, under 'bistable' and 'monostable' conditions on the nonlinear function $f : \mathbb{R}^N \to \mathbb{R}^N$, when A is a positive-definite diagonal matrix. Here we observe that this genre of hypotheses on f yields results on the existence and properties of intermediate zeros of f between S and T. These imply the existence of a 'chain' of monotone travelling waves from S to T under 'unstable' conditions on f for which non-existence of a direct monotone connection had been established. We also prove that for wave velocities consistent with the known existence theory, there are unstable and stable monotone directions at S and T respectively. Further, the dimensions of the unstable manifold at S and the stable manifold at T sum to at least $2N$, the dimension of the underlying phase space.

We then address the question of travelling-wave solutions to parabolic systems of the form

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + G(u, \frac{\partial u}{\partial x})\frac{\partial u}{\partial x} + f(u),$$

in which the nonlinearity depends on the gradient of the dependent variable. Here f satisfies the *'bistable'* conditions, and G is a nonlinear matrix-valued function satisfying certain growth conditions. Such gradient-dependent terms are called convection terms. The existence of a monotone travelling-wave solution connecting two equilibria is shown for such systems using a homotopy degree argument together with the known existence result for the convectionless system.

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Contents

Chapter 1

Introduction

Systems of nonlinear parabolic partial differential equations arise in the mathematical modelling of a vast number of disparate areas of physical and biological science, such as population genetics, chemical kinetics and nerve axon theory. Those from the important class of *reaction-diffusion* systems may be written in the form

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(u), \qquad u \in \mathbb{R}^N, x \in \mathbb{R}, t \in [0, \infty). \tag{1.1}$$

Here A is a real, positive-definite $N \times N$ matrix, and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuously differentiable nonlinear function. The vector u may represent, for example, the concentrations of a number of chemicals, or the population densities of interacting species. The interaction between components of u is modelled by the reaction term, $f(u)$. So $f(u)$ could arise from the effect on concentrations of chemical reaction, or from the effect of interspecies relationships on population densities. Random movement in the physical system is modelled by the *diffusion* terms, $A\frac{\partial^2 u}{\partial x^2}$.

Travelling waves are solutions u of (1.1) in the form

$$u(x,t) = v(x - ct). \tag{1.2}$$

Here $v : \mathbb{R} \to \mathbb{R}^N$ is the profile of the wave which propagates through the one-dimensional spatial domain at constant velocity c. Clearly, v satisfies the system of ordinary differential equations

$$Av'' + cv' + f(v) = 0. \tag{1.3}$$

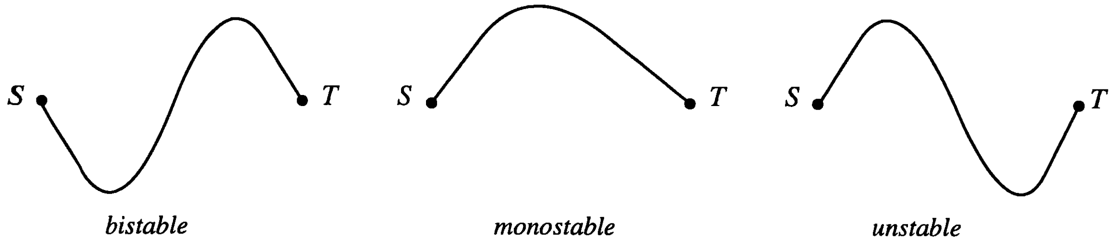


figure 1 : Different forms of f in the one-dimensional case.

The existence of solutions to (1.3) depends strongly on the nature of the non-linearity f. In particular, it depends on the form of f near equilibria. (An *equilibrium* of f is a point $w_0 \in \mathbb{R}^N$ where $f = 0$.) We will classify equilibria to reflect the stability of the equation $\frac{du}{dt} = f(u)$. An equilibrium of f is said to be *stable* if all the eigenvalues of $df[w_0]$ are in the open left-half complex plane, and *unstable* if there is an eigenvalue of $df[w_0]$ in the open right-half plane. ($df[w_0]$ denotes the Fréchet derivative of f at w_0, or its Jacobian matrix, the matrix of partial derivatives of f at w_0 with respect to the standard basis of $\mathbb{R}^N$, as appropriate.)

Now let $S, T \in \mathbb{R}^N$ be equilibria of f, such that componentwise, $S_k < T_k$ for each $k \in \{1, \ldots, N\}$. Then system (1.1) is called *bistable* if S and T are both stable equilibria of f, *monostable* if only one is stable, and *unstable* if neither is stable. When $N = 1$ and f is a scalar-valued function, these conditions are equivalent to specifying the gradient of f at S and at T (see figure 1). Such f arise in biology, and in combustion theory in chemical physics.

Our primary interest is the existence of monotone solutions of (1.3) connecting the equilibria S (source) and T (target). Precisely, we seek heteroclinic orbits $\{v(s) : s \in \mathbb{R}\}$ of (1.3), where the component function v_k is a monotone real-valued function asymptotic to S_k and T_k as $s \to -\infty$ and $s \to \infty$ respectively. Such solutions, called *travelling fronts* or sometimes *monotone connections* - see figure 2, correspond to smooth transitions between steady states of the physical systems being modelled by (1.1). Their form is consistent with the fact that the components of v usually model bounded non-negative quantities (such as concentrations, for example). They are interesting not only as an example of behaviour widely exhibited in nature, but also because they sometimes appear in the characterisation of the asymptotic behaviour of the general initial value prob-

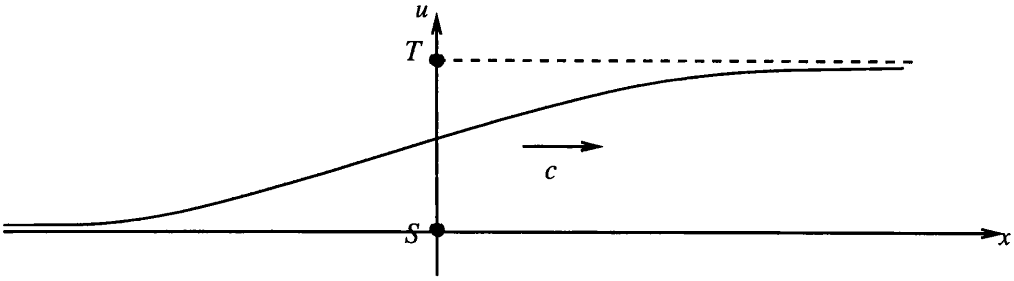


figure 2 : Profile of a travelling-front solution to (1.1) at a fixed time t. For a positive wave velocity c, the profile moves to the right as time increases from t.

lem of (1.1). Clearly, the study of travelling waves encroaches on both the theory of partial differential equations and of finite and infinite-dimensional dynamical systems.

Travelling-wave solutions to the scalar equation (1.1) ($N = 1$) have been extensively studied. The first paper to consider such propagating waves mathematically was the seminal work of Kolmogorov, Petrovsky and Piskounov [28] in 1937, which arose from a study for the propagation of a dominant gene in mathematical biology. Fisher [21] published the same model in the same year. There $f(u) = u(1 - u)$, so the nonlinearity is *monostable.* Existence of a travelling front solution to (1.1) is shown in [28] for all velocities above a positive critical value. The stability of these solutions is also studied, together with the general initial value problem. Other important early work is that of Zeldovich and Frank-Kamenetskii [48], who considered a combustion model in 1940. In the 1960s, Kanel [25] proved the existence of unique (*modulo* translations) travelling front for the bistable equation in the case where f has exactly one intermediate equilibrium between S and T. Aronson and Weinberger [2] also treated this particular bistable case, motivated this time by genetics. Perhaps the most general study of the scalar bistable equation has been that of Fife and McLeod [20], who in 1977 established stability properties and treated the general problem where there is more than one intermediate equilibrium. Probabilists, for example, McKean [31], have also studied these equations intensively. The books by Britton [5] and Fife [19] are important references and Murray [35] gives valuable insight into the biological motivation for addressing these questions.

That the scalar equation has been so intensively studied is firstly because parabolic equations admit a comparison principle, and secondly because the phase space analysis of the ordinary differential equation (1.3) is planar. For *systems* of equations, the phase space becomes much more difficult to analyse, and there is no general comparison principle. In consequence, work on systems is far less complete than that for the scalar equation. Methods involving phase space and 'shooting' type arguments have been effective for some specific systems of two coupled equations. For example, [8] treats a generalisation of the equation in [28] both by an analytic 'shooting' technique, and by probability theory. Conley index theory has been used, notably by Conley, Gardner, Mischaikow, Reineck and Hutson [33, 34, 32, 22] to prove the existence of travelling waves for systems with small numbers of equations.

For systems with an arbitrary number of equations, significant work has been done by A.I.Vol'pert, V.A. Vol'pert and V.A. Vol'pert [42, 44, 43]. Attention there is restricted to continuously differentiable nonlinearities f satisfying the so-called *local monotonicity* property: whenever $u \in \mathbb{R}^N$ is such that $f_i(u) = 0$ for some i, $1 \leq i \leq N$, then

$$\frac{\partial f_i(u)}{\partial u_j} > 0 \text{ for } j \neq i, \quad j = 1, \ldots, N. \tag{1.4}$$

where f_i denotes the i^{th} component function of f. When f satisfies (1.4) for *every* $u \in \mathbb{R}^N$ and *every* $i \in \{1, \ldots, N\}$, f is said to satisfy the *global monotonicity* property. These properties are the natural hypotheses for mutualistic interactions in biology, and for certain kinds of chemical kinetics. They are of great mathematical importance. Firstly, systems (1.1) where f satisfies the global monotonicity property *do* admit a comparison principle - see, for example, [38]. The method of upper- and lower- solutions can thus be used in [44] to show the existence of monotone solutions to (1.3) in the monostable case if and only if the velocity c exceeds a positive critical value (*cf* the existence results of [28]). That paper also shows that there are no monotone connections in the unstable case. Secondly, these monotonicity properties allow exploitation of Perron-Frobenius theory. If $w_0 \in \mathbb{R}^N$ is such that $\dfrac{\partial f_i}{\partial u_j}(w_0) > 0$ for $i \neq j$, $i, j = 1, \ldots, N$, then the off-diagonal elements of $df[w_0]$ are positive - that is, $df[w_0]$ is a Perron-Frobenius matrix. Hence if f satisfies the local monotonicity property (1.4), then $df[u]$ is a Perron-Frobenius matrix for each $u \in \mathbb{R}^N$ with $f(u) = 0$. We will capitalise on this last observation throughout this thesis.

In [42], the Vol'perts treat the locally monotone bistable case in which there are no intermediate equilibria which are stable. ($E \in \mathbb{R}^N$ is said to be an *intermediate equilibrium* of f if $f(E) = 0$ and $S \leq E \leq T$, where $\leq$ is the standard ordering on $\mathbb{R}^N$ induced by the positive quadrant.) It is this paper that was the stimulus for much of the work of this thesis, and hence we describe its ideas in some detail. Their approach is to use a degree theory. Very loosely, degree theory assigns an integer called the degree to a given function defined on some set; it gives an 'algebraic' count of the number of zeros of the function in this set. A degree has the property that if it is non-zero, there must be a zero of the function in the set. The degree is also known to be invariant under a class of perturbations of the function under which no zeros of the function 'escape' from the set - this perturbation is known as a homotopy. Whence existence of solutions to a given equation in a specified domain can be proved by showing that the given function is a deformation of a simpler one for which the degree is known to be non-zero. (This general technique is sometimes called the Leray-Schauder principle.)

The first task is to find a degree theory appropriate for the problem. Prototype degree theories are the Brouwer degree (1912, [6]), defined for continuous functions $\phi : \mathbb{R}^N \to \mathbb{R}^N$, and the Leray-Schauder degree (1934, [30]), defined for compact perturbations of the identity acting from a Banach space to itself. If (1.3) were considered on a *bounded* domain (specifying suitable boundary conditions), then a corresponding operator equation could be reduced to the form $I - K$, where K is compact. But on the unbounded domain $\mathbb{R}$, this fails. Sometimes, compact operators can be regained by setting up the operator equation in a Sobolev space with a weighted inner product, where the weight grows more quickly than exponentially at infinity - see, for example, [18]. However, since solutions of (1.3) that approach a hyperbolic equilibrium will do so at an exponential rate, they do not belong to such spaces. In [42] , the Vol'perts consider an operator equation associated with (1.3) in a weighted-Sobolev-space in which the weight only grows algebraically at infinity. The corresponding operators are then shown to be of a type known as $(S)_+$, for which a degree theory was developed in the seventies by Browder [7] and Skrypnik [40]. Essentially, $(S)_+$ operators are of the form 'monotone plus compact'; the fact that the system is *bistable* yields the monotone term, and the remainder is compact because of the use of weighted spaces in which the weights grow at infinity. (This degree theory and weighted space will be discussed in detail in Chapter 5.)

The translation invariance of (1.3) is used to overcome the fact that the wave velocity c is also *a priori* unknown. For each $h \in \mathbb{R}$, $v_h(s) := v(s+h)$, $s \in \mathbb{R}$ solves (1.3) whenever v does. In [42], the parameter c is 'functionalised'; a functional is constructed on the weighted Sobolev space so that for each fixed element of the space, the functional is monotonic and surjective in the translation h. The system (1.3) with the parameter c replaced by this functional is equivalent to the original system.

The existence of a monotone travelling wave is shown by constructing a homotopy from the original system to a system which is basically N-copies of a scalar bistable equation with one intermediate equilibrium. The degree of this latter system is calculated to be $+1$ using the existence and uniqueness results known for travelling-wave solutions of the scalar equation. The homotopy invariance of degree then yields existence for the original system. The fact that the solution is monotone is obtained from the construction of the set on which the homotopy is considered. The set is formed by first proving weighted Sobolev space *a priori* bounds on all monotone solutions at any stage of the homotopy. This critically uses the hypothesis on the absence of intermediate stable equilibria, near which a solution could 'linger', and an interplay between the functionalisation of the velocity and the weighted function space. It is then proved that any non-monotone solution must be uniformly bounded away from any monotone solution in the Sobolev space. Here, the local monotonicity property of f, together with the bistable condition, are vital. Together, these bounds yield an appropriate set in the weighted Sobolev space, which is known *a priori* to contain no non-monotone solutions. In fact, when f satisfies the *global* monotonicity property, [42] also shows uniqueness (*modulo* translations) of the monotone travelling wave. They prove that any monotone solution must have associated index $+1$. Since the overall degree is $+1$ and degree is additive, uniqueness follows. Recently, the Vol'perts extended this work to consider the case when there are intermediate stable equilibria, and obtain results analogous to those of Fife and McLeod [20] for the scalar equation -see [45].

This thesis is devoted to further study of the systems considered by the Vol'perts. The global operator theoretic nature of the Vol'perts techniques rather obscures the dynamical systems context of (1.3). Setting $v = w'$, the second order system of N ordinary differential equations (1.3) becomes a first order system of $2N$ equations,

$$\begin{pmatrix} w' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ -cA^{-1}v - A^{-1}f(w) \end{pmatrix} \tag{1.5}$$

whose linearization about $(v, w) = (0, w_0)$ with $f(w_0) = 0$ is

$$\begin{pmatrix} w' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A^{-1}B & -cA^{-1} \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} =: K_c(w_0) \begin{pmatrix} w \\ v \end{pmatrix} \tag{1.6}$$

where $B = df[w_0]$. Recall that in general, the behaviour of a nonlinear system at a hyperbolic equilibrium is determined by the eigenvalue structure of its linearization there. The dimensions of the unstable and stable manifolds about a hyperbolic equilibrium point w_0 (denoted by $D_u(w_0)$ and $D_s(w_0)$, say) equal the dimensions of the unstable and stable subspaces of the linearized problem (1.6), which in turn are equal to the numbers of eigenvalues (counted according to algebraic multiplicity) of $K_c(w_0)$ in the open right- and left-half planes respectively - see, for example, [9, 37]. So upon manipulating (1.6), we see that the qualitative properties of the system in the vicinity of the equilibria at S and T can be gleaned from the 'eigenvalues' $\lambda \in \mathbb{C}$ and corresponding $z \in \mathbb{C}^N \backslash \{0\}$ of the nonlinear-in-λ, linear eigenvalue problem

$$(\lambda^2 A + \lambda c I + B)z = 0 \tag{1.7}$$

where B is the derivative of f at the equilibrium point w_0. (Note that these values λ are eigenvalues of $K_c(w_0)$ in the usual sense, with eigenvector $\begin{pmatrix} z \\ \lambda z \end{pmatrix} \in \mathbb{C}^{2N} \backslash \{0\}$.)

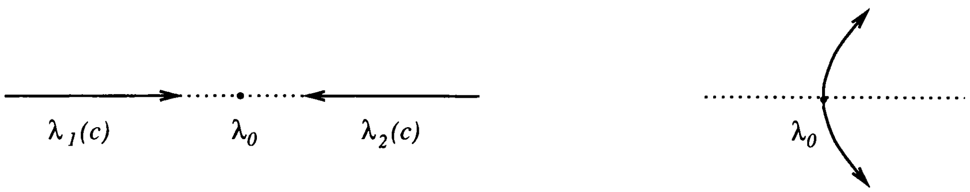
For a given velocity c, such λ and z will henceforth be referred to as *eigenvalues and eigenvectors of the travelling-wave problem linearized at* w_0. The linear space of all $z \in \mathbb{C}^N$ such that $(\lambda^2 A + \lambda c I + B)z = 0$ will be called the *eigenspace* corresponding to the travelling-wave eigenvalue λ. By the *stable (unstable) manifold at an equilibrium* w_0, we will mean the stable (unstable) manifold of the travelling-wave system (1.5) at w_0. **(This use of stable/unstable should not be confused with the notion of the stability of an *equilibrium* introduced previously.)** For brevity, we will write $M(\lambda, c) := \lambda^2 A + \lambda c I + B$, where the matrices A and B are fixed in a given context. An eigenvalue λ of the travelling-wave problem linearized at an equilibrium w_0 with corresponding eigenvector z, is *stable (unstable) monotone* if

(i) λ is real and negative (positive), and

(ii) all components of the vector z are of the same sign.

There is said to be a *stable (unstable) monotone direction* at w_0 if there is a stable (unstable) monotone eigenvalue at w_0. Such eigenvalues are of pivotal importance. In Chapter 2, we establish that for the existence of a monotonic connection satisfying a general form of (1.3), it is *necessary* that there exist unstable and stable monotone eigenvalues of the travelling-wave problem linearized at S and T respectively. This preliminary chapter also contains notation and some positive operator theory.

Chapter 3 is then devoted to proving results about eigenvalues of the travelling-wave problem and their dependence on the wave velocity c. For it to be feasible to prove that there exists a travelling wave from S to T for a given value of the velocity c, we expect that $D_u(S) + D_s(T) \geq 2N$, the dimension of the underlying phase space, and $D_u(S) + D_s(T) \geq 2N + 1$ is necessary for the unstable manifold at S and the stable manifold at T to intersect transversally. In Chapter 3, we show that this dimensional analysis ties in with the results of Vol'pert and Vol'pert. Decisive use is made of a result due to Cohen [10] on the convex dependence of the Perron-Frobenius eigenvalue on the diagonal of a Perron-Frobenius matrix. In the bistable case, we show that there exist both stable and unstable monotone eigenvalues at S and at T and $D_s(S) = D_s(T) = D_u(S) = D_u(T) = N$, for all velocities c. It is shown in [42] that there is a *unique* c and corresponding travelling-wave w *unique* modulo translations satisfying (1.3) in this case, so it is not surprising that the dimensions of the invariant manifolds only just 'add up' since the solution does not persist under perturbation of the wave velocity. In the monostable case in which S and T are stable and unstable equilibria respectively, Vol'pert and Vol'pert prove the existence of monotone travelling waves for all



(a) As c is decreased to c_0, $\lambda_1(c)$ and $\lambda_2(c)$ both tend monotonically to λ_0.

(b) As c is decreased below c_0, a pair of complex conjugate branches of eigenvalues emanate from λ_0.

figure 3 : The dependence of eigenvalues on c near c_0

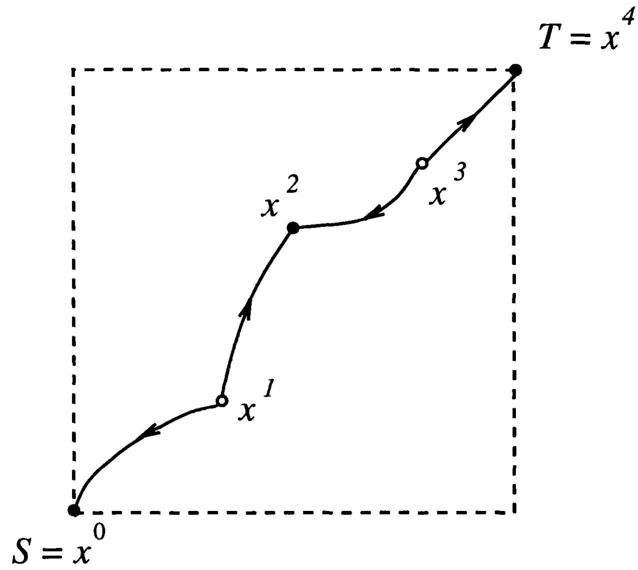
$c \geq \tilde{c} > 0$. We show that there exists some $c_0 > 0$ such that there is a unique stable monotone eigenvalue λ_0 at T for $c = c_0$, and that for $c > c_0$, there are *two* such eigenvalues, $\lambda_1(c) < \lambda_0 < \lambda_2(c)$. Moreover, λ_1 and λ_2 are decreasing and increasing continuous functions of c respectively, each of which tends to λ_0 as c is decreased to c_0. For $c < c_0$, there is *no* stable monotone eigenvalue and a pair of complex conjugate branches of eigenvalues emanates from λ_0. (See figure 3.) Using the Cohen convexity theorem, it is proved that for $c > c_0$, there can be no eigenvalues with real part between $\lambda_1(c)$ and $\lambda_2(c)$. Together with the observation that N eigenvalues tend to $-\infty$ as $c \to \infty$, this shows that for such c, $D_s(T) \geq N+1$ and hence $D_s(T) + D_u(S) \geq 2N+1$. (The structure at S follows from the bistable analysis.) The fact that we have a surplus of dimensions here is compatible with the Vol'perts' result that there is a continuum of velocities for which monotone solutions exist.

Replacing λ and c in (1.7) by their negatives, it is clear that when the eigenvalue conditions on $df[S]$ and $df[T]$ are interchanged, there exists some $c_0 < 0$ such that there is an unstable monotone eigenvalue at S precisely when $c \leq c_0$, and that for such velocities, $D_s(T) + D_u(S) \geq 2N+1$. Hence if *both* S and T are unstable equilibria, there is no velocity such that both an unstable monotone direction at S and a stable monotone direction at T exist. Using the result of Chapter 2, this establishes that there cannot be a monotone connection from S to T in the unstable case. No claims are made about the existence or behaviour of centre manifolds. Numerical experiments have shown that there can be eigenvalues of the travelling-wave problem on the imaginary axis in the monostable case, but we are concerned simply with counting the number of eigenvalues in the open left-half plane, which give the dimension of the stable manifold, and the number of eigenvalues in the open right-half plane, which give the dimension of the unstable manifold.

In Chapter 4, we turn to a study of the existence of equilibria of f in addition to S and T. The hypothesis in the main existence theorem in the bistable case [42], is that there exists at least one equilibrium of f between S and T (an intermediate equilibrium), and that each such equilibrium is unstable. The purpose of this assumption is to ensure that there is no intermediate stable equilibrium of f. For *scalar-valued* differentiable functions f, it is clearly true that the bistable conditions imply the existence of an intermediate zero of f, and it is germane to ask the following — do the eigenvalue conditions on $df[S]$ and $df[T]$ in the bistable case imply the existence of zeros of f between those at S and T for an N-dimensional system? The answer to this question is yes. We prove that given a continuously differentiable locally monotone function $f : \mathbb{R}^N \to \mathbb{R}^N$ with stable equilibria at S and T, there exists x between S and T with $f(x) = 0$. Moreover, there is also an intermediate equilibrium in the case when both S and T are *unstable* equilibria of f. The proofs use positive operator theory to exploit the monotonicity assumptions on f, which together with the eigenvalue conditions allow existence to be shown using Brouwer degree.

When the zeros of f are isolated and the Fréchet derivative of f at each zero does not have zero as its eigenvalue of largest real part, even more can be said. If both S and T are stable (unstable) equilibria, there is an unstable (stable) equilibrium of f in the order interval between S and T. In fact, given a maximal (in the sense of set inclusion) totally-ordered set of equilibria in this interval, $\{x^1, \ldots, x^k\}$, k must be odd and x^i is a stable or unstable (unstable or stable) equilibrium according to whether i is even or odd. (A totally-ordered set of equilibria is a set of equilibria such that $x \leq y$ or $y \leq x$ for each x, y in the set.)

This has important consequences for the 'unstable' case of Vol'pert and Vol'pert (that is, when S and T are both unstable equilibria). Although no monotone connection from S to T can exist, the alternating stability of the equilibria $\{x^1, \ldots, x^k\}$ enables the existence theory for the 'monostable' case [44] to be applied to the order interval between x^i and x^{i+1} for each i, $0 \leq i \leq k$. Thus there *are* monotone connections from S and T to the intermediate zeros x^1 and x^k respectively, and also between x^i and x^{i+1} for $1 \leq i \leq k$ if $k > 1$. For example,



An example of multiple connections in the unstable problem, in the case when n=2 and there are 3 intermediate equilibria as shown. • and ∘ denote unstable and stable equilibria respectively. For sufficiently large positive velocities, there are monotone connections between adjacent equilibria in the directions indicated by the arrows.

figure 4

if there is exactly one zero x of f between S and T, then for sufficiently large positive velocities, there is an increasing monotone connection from x to T (that is, $w'(x) > 0$ for $x \in \mathbb{R}$) and a decreasing monotone connection from x to S ($w'(x) < 0$ for $x \in \mathbb{R}$). (See figure 4.) Note that Dancer ([12, 13]) has proved related results on the existence and stability of intermediate equilibria for a type of discrete monotone mappings. The material of Chapters 3 and 4 appears in [11].

Chapters 5-7 are concerned with an extension of the degree theoretic ideas used in the bistable case [42] to systems in which the nonlinearity depends on the gradient of the dependent variable. Such gradient dependences arise in applications in which there is an underlying drift; for example, chromatography, convection in chemical reactions or wind effects in biology - see Murray [35] for details. Some work has been done on scalar-valued gradient-dependent problems, references to which are contained in [43]. For *systems*, however, we know of no prior published work. We prove an existence theorem, analogous to that in [42], for the bistable system

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + G(u, \frac{\partial u}{\partial x})\frac{\partial u}{\partial x} + f(u) \tag{1.8}$$

where A and f are as in [42], and $G : \mathbb{R}^N \times \mathbb{R}^N \to M^{N\times N}$ is a continuously differentiable function satisfying

(G1) G is diagonal-matrix valued;

(G2) there exist continuous functions $\beta, \gamma : \mathbb{R}^N \to \mathbb{R}_+$ such that for each $u, v \in \mathbb{R}^N$,

$$||G(u,v)v|| \leq \beta(u) + \gamma(u)||v||,$$

where $||\cdot||$ denotes a norm on $\mathbb{R}^N$;

(G3) $G(u, 0) = 0$ for each $u \in \mathbb{R}^N$ with $f(u) = 0$.

(The system (1.8) is said to be *bistable* when the system with $G \equiv 0$ is bistable.)

We use degree theory. The idea of our approach is to homotope from the gradient-dependent system (1.8) to the system with $G \equiv 0$ treated in [42], with the aim of deducing the existence of a monotone travelling-wave solution of (1.8) from the fact that the degree related to (1.1) is known from [42] to be non-zero.

Problems arise in implementing this. The operators associated with (1.8) in analogy with (1.1) are no longer clearly in the domain of definition of the degree function for $(S)_+$ operators. This difficulty is overcome by approximating, and then taking a limit to recover the original problem. Restricting to travelling-wave solutions of (1.8) leads to the system

$$Av'' + cv' + G(v, v')v' + f(v) = 0 \tag{1.9}$$

where c is an unknown constant as before. We first consider the modified system

$$Av''(s) + cv'(s) + \sigma_R(s)G(v(s), v'(s))v'(s) + f(v(s)) = 0, \ \ s \in \mathbb{R} \tag{1.10}$$

where for $R > 0$, $\sigma_R \in C^\infty(\mathbb{R}, [0, 1])$ is supported in $[-R-1, R+1]$, with $\sigma_R(s) = 1$ for $s \in [-R, R]$. For each $R > 0$, the existence of a monotone solution to (1.10) asymptotic to S and T is proved using $(S)_+$ degree theory via a homotopy to (1.3). Such a solution to (1.9) is then obtained by taking the limit of these solutions as $R \to \infty$.

Chapter 5 begins with some technical material on weighted function spaces and $(S)_+$ degree theory. We then formulate the approximate problem (1.10) in a weighted Sobolev space. The technique of functionalisation of the velocity (mentioned earlier in this introduction) is developed. The remainder of the chapter is concerned with showing that for each $R > 0$, an operator associated with (1.10), acting in the weighted function space, lies in the domain of definition of $(S)_+$ degree. The compact support of σ_R, together with condition **(G2)**, are vital in giving the compactness of the contribution of the "$\sigma_R G(v, v')v'$ " term. (Recall from above that, roughly speaking, $(S)_+$ operators are of the form ' monotone + compact'.)

Chapter 6 is concerned with uniform *a priori* estimates for monotone solutions of the approximate system (1.10). The ultimate aim of this chapter is to prove that *independent* of R, the difference between a monotone solution of (1.10) and a fixed C^∞ function ψ lies in a fixed bounded set in the weighted Sobolev space in which (1.10) is formulated. We first prove some estimates for a generalisation of (1.10) in which σ_R is replaced by σ, a smooth function, bounded between 0 and 1 but not necessarily of compact support. The bounds obtained are *independent* of the choice of σ satisfying these conditions. This independence eventually yields the required Sobolev space bound. Conditions **(G1)** and **(G2)** come into play directly. Vital use is also made of the necessary condition for the existence of monotone connections proved in Chapter 2, which is the *raison d'être* of condition **(G3)**.

The existence results for the class of gradient-dependent systems are the subject of Chapter 7. First we fix $R > 0$. The existence of a monotone solution to (1.10) asymptotic to S and T is obtained using a homotopy degree theory argument as discussed above. Then the uniform Sobolev space bound from Chapter 6 gives the existence of a weak limit u of a sequence $\{u_n\}$ of monotone solutions of (1.10), u_n corresponding to $R = n \in \mathbb{N}$. We show that $v = u + \psi$ is a monotone solution of (1.9). Constructing the limiting solution in this way ensures that the asymptotic boundary conditions are satisfied - that is, $v(s) \to S, T$ as $s \to -, +\infty$.

Chapter 2

Preliminaries

This chapter is concerned with preliminary material needed throughout this thesis. We begin with some notation, followed with some basic results from positive operator theory. The main part of the chapter is devoted to the proof of a vital *necessary* condition for the existence of monotone travelling fronts.

2.1 Notation

The following conventions will be adhered to throughout. We use the notation ':=' to mean 'is defined to equal'. Denote the usual inner products and corresponding norms on $\mathbb{R}^N$ and $\mathbb{C}^N$ by $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$, $\|\cdot\|_{\mathbb{C}^N}$ respectively.

For vectors $p, q \in \mathbb{R}^N$, we say that $p < q$ $(p \leq q)$ if $p_k < q_k$ $(p_k \leq q_k)$ for each k, $1 \leq k \leq N$ and p_k denotes the k^{th} coordinate of p. When $q < p$, (q, p) is the set $\{x \in \mathbb{R}^N : q < x < p\}$ and $[q, p]$ is the set $\{x \in \mathbb{R}^N : q \leq x \leq p\}$. A vector $x \in \mathbb{R}^N$ is *positive* (*non-negative*) if all its components are positive (non-negative). We write $x > 0$ $(x \geq 0)$. We denote the positive cone in $\mathbb{R}^N$ of non-negative vectors by $\mathbb{R}^N_+$; when $N = 1$, $\mathbb{R}^1_+$ is abbreviated to $\mathbb{R}_+$ (that is, $\mathbb{R}_+ := [0, \infty)$). The set of all real $N \times N$ matrices with strictly positive off-diagonal elements is denoted by $P^{N \times N}$. The set of all real $N \times N$ matrices is denoted as usual by $M^{N \times N}$. The entry in the $i-th$ row and $j-th$ column of a matrix M is denoted by M_{ij}. If $M \in M^{N \times N}$ is diagonal, then we use the shorthand M_i for the $i-th$ element on the diagonal of M. Throughout, A denotes a real, positive-definite diagonal matrix.

For a Fréchet differentiable function $f : \mathbb{R}^N \to \mathbb{R}^N$, we denote the Fréchet derivative of f at $w \in \mathbb{R}^N$ by $df[w]$. Recall that $df[w]$ can be identified with its Jacobian matrix, the matrix of partial derivatives of f at w with respect to the standard basis of $\mathbb{R}^N$. Without further comment, we will use the notation $df[w]$ to denote the derivative or its Jacobian matrix, as appropriate.

Let Ω be an open subset of a real finite-dimensional space (for example, $\mathbb{R}, \mathbb{R}^N, M^{N\times N}, \mathbb{R}^N \times \mathbb{R}^N$). Denote the closure of Ω by $\overline{\Omega}$, the boundary by $\partial\Omega$ and the interior by Ω^o. Let Y be a subset of a (possibly different) real finite-dimensional space. Then for $k \in \mathbb{N}$, we write $C^k(\Omega, Y)$ for the linear space of functions $f : \Omega \to Y$ which, together with all the partial derivatives of each component function of f of order at most k are continuous on Ω. Let $C^\infty(\Omega, Y) = \cap_{k=0}^{\infty} C^k(\Omega, Y)$, and denote by $C_0^\infty(\Omega, Y)$ the subspace of $C^\infty(\Omega, Y)$ consisting of those functions with compact support in Ω.

Let $C^k(\overline{\Omega}, Y)$ be comprised of those $f \in C^k(\Omega, Y)$ for which all the partial derivatives of each component function of f of order at most k are bounded and uniformly continuous on Ω. Then $C^k(\overline{\Omega}, Y)$ is a Banach space with norm

$$||f||_{C^k(\overline{\Omega},Y)} = \Sigma_{i,|\alpha|\leq k} \sup_{x\in\Omega} |D^\alpha f_i| \tag{2.1}$$

where $D^\alpha f_i$ denotes the partial derivative of the i^{th} component of f given by the multi-index α. Note that we may suppress explicit mention of the set Y if it is clear from the context.

Now let Ω be an open subset of $\mathbb{R}$. Let $L_\infty(\Omega, \mathbb{R}^N)$ comprise of equivalence classes of measurable functions $u : \Omega \to \mathbb{R}^N$ that are essentially bounded; this space is endowed with the norm

$$||u||_{L_\infty(\Omega,\mathbb{R}^N)} = \text{ ess sup}_{x\in\Omega}||u(x)||, \tag{2.2}$$

Denote by $L_2(\Omega, \mathbb{R}^N)$ the linear space of equivalence classes of measurable functions $u : \Omega \to \mathbb{R}^N$ such that

$$||u||_{L_2(\Omega,\mathbb{R}^N)} := \langle u, u\rangle_{L_2(\Omega,\mathbb{R}^N)} < \infty, \tag{2.3}$$

where for u, v measurable, the inner product is given by

$$\langle u, v\rangle_{L_2(\Omega,\mathbb{R}^N)} := \int_\Omega \langle u(s), v(s)\rangle ds. \tag{2.4}$$

We write $W_2^1(\Omega, \mathbb{R}^N)$ for the subset of $L_2(\Omega, \mathbb{R}^N)$ consisting of functions $u \in L_2(\Omega, \mathbb{R}^N)$ for which $\mathcal{D}u \in L_2(\Omega, \mathbb{R}^N)$, where $\mathcal{D}u$ is the vector of weak derivatives of the components of u. Recall from Adams [1] that the Sobolev space $W_2^1(\Omega, \mathbb{R}^N)$ is a separable Hilbert space, with inner product

$$\langle u, v \rangle_{W_2^1(\Omega,\mathbb{R}^N)} := \langle u, v \rangle_{L_2(\Omega,\mathbb{R}^N)} + \langle \mathcal{D}u, \mathcal{D}v \rangle_{L_2(\Omega,\mathbb{R}^N)} \quad (u, v \in W_2^1(\Omega, \mathbb{R}^N)) \tag{2.5}$$

and norm $\|\cdot\|_{W_2^1(\Omega,\mathbb{R}^N)}$, and $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in $W_2^1(\mathbb{R}, \mathbb{R}^N)$. If Ω is clear from the context, then we will suppress explicit mention of it in the subscripts on $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ and write $\langle \cdot, \cdot \rangle_{L_2}$, etcetera.

The spaces of complex-valued functions $\tilde{L}_2(\mathbb{R}, \mathbb{C}^N)$, $\tilde{W}_2^1(\mathbb{R}, \mathbb{C}^N)$, $\tilde{C}_0^\infty(\mathbb{R}, \mathbb{C}^N)$ will occasionally be required. A function $u : \mathbb{R} \to \mathbb{C}^N$ belongs to $\tilde{L}_2(\mathbb{R}, \mathbb{C}^N)$ if

$$\int_{\mathbb{R}} \langle u(s), u(s) \rangle_{\mathbb{C}^N} \, ds < \infty. \tag{2.6}$$

$u : \mathbb{R} \to \mathbb{C}^N$ belongs to $\tilde{W}_2^1(\mathbb{R}, \mathbb{C}^N)$ if $u, \mathcal{D}u \in \tilde{L}_2(\mathbb{R}, \mathbb{C}^N)$. $\tilde{C}_0^\infty(\mathbb{R}, \mathbb{C}^N)$ is comprised of functions $u : \mathbb{R} \to \mathbb{C}^N$ whose real and imaginary parts belong to $\tilde{C}_0^\infty(\mathbb{R}, \mathbb{R}^N)$.

2.2 Positive operator theory

Recall from Chapter 1 the vital connection between Perron-Frobenius theory and a function $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfying a monotonicity property. This observation underlies much of the work of this thesis. We will exploit the following well-known version of the Perron-Frobenius theorem (see, for example Seneta [39]).

Theorem 2.2.1 *(Perron-Frobenius). Let $M \in P^{N \times N}$. Then M has a real, simple eigenvalue $\mu_{PF}(M)$ such that an associated eigenvector is positive , and every other eigenvalue of M has real part less than $\mu_{PF}(M)$. Moreover, any positive eigenvector must be a multiple of the eigenvector corresponding to $\mu_{PF}(M)$. We call $\mu_{PF}(M)$ the Perron-Frobenius eigenvalue of M, and the associated positive eigenvector x with $\|x\| = 1$ the Perron-Frobenius eigenvector of M.*

Such eigenvalues satisfy the following property - see, for example, Varga [41].

Theorem 2.2.2 *Increasing (decreasing) elements of a matrix $M \in P^{N \times N}$ increases (decreases) $\mu_{PF}(M)$.*

To utilise Theorem 2.2.1 to the full, we will need a special case of a theorem due to Krasnosel'skii [29].

Theorem 2.2.3 *(Krasnosel'skii). Let M be a real $N \times N$ matrix with non-negative entries such that $M^p u \geq \alpha u$ $(\alpha > 0)$ for some $p \in \mathbb{N}$ with $u \in \mathbb{R}^N \backslash \{0\}$, $-u \notin \mathbb{R}^N_+$. Then M has a real eigenvalue λ which satisfies $\lambda \geq \sqrt[p]{\alpha} > 0$.*

The following corollary is particularly useful.

Corollary 2.2.4 *Let $M \in P^{N\times N}$ and suppose that there exists $u \in \mathbb{R}^N \backslash \{0\}$ with $-u \notin \mathbb{R}^N_+$ such that $Mu > 0$. Then $\mu_{PF}(M) > 0$.*

Proof. Since $Mu > 0$, there exists some $\beta > 0$ such that $Mu > \beta u$. Also, there exists $\alpha > 0$ such that $M + \alpha I$ has positive entries, and $(M + \alpha I)u > (\alpha + \beta)u$, so by Theorem 2.2.3 , $\mu_{PF}(M + \alpha I) \geq \alpha + \beta$. Then, clearly, $\mu_{PF}(M) = \mu_{PF}(M + \alpha I) - \alpha \geq \beta > 0$.

□

We will also need a related result, which is a variant of Theorem 1.6 in Seneta [39].

Theorem 2.2.5 *Let $M \in P^{N\times N}$ and suppose that there exists $u \in \mathbb{R}^N_+ \backslash \{0\}$ such that $Mu < 0$. Then $\mu_{PF}(M) < 0$.*

2.3 Existence of monotone directions

We prove a necessary condition for the existence of monotone directions at an equilibrium. The purpose is twofold. Firstly, this result motivates the material of Chapter 3, in which the existence of monotone directions for equilibria of different stabilities is analysed as a function of the wave velocity c. It also plays a vital role in the *a priori* bounds required for the degree theoretic proof of existence of monotone travelling waves for the gradient-dependent problem (1.8) (see chapter 6). For reasons related to the problem of obtaining these estimates, we treat a fairly general form of travelling-wave problem. Here, we require that the gradient-dependent nonlinearity $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ satisfy condition **(G3)**. The idea for the following proof is reasonably classical, but we have been unable to find a statement with this level of precision in $\mathbb{R}^N$, for arbitrary N, in the non-autonomous case needed here.

Theorem 2.3.1 *Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $w_0 \in \mathbb{R}^N$ be such that $f(w_0) = 0, df[w_0] \in P^{N\times N}$ and $\mu_{PF}(df[w_0]) \neq 0$. Let $A \in M^{N\times N}$ be a positive-definite diagonal matrix, and $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ satisfy condition* **(G3)**. *Let $\sigma : \mathbb{R} \to [0,1]$. Suppose that $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies*

$$Aw''(s) + cw'(s) + \sigma(s)G(w(s), w'(s))w'(s) + f(w(s)) = 0, \ s \in \mathbb{R} \tag{2.7}$$

for some $c \in \mathbb{R}$. Further, suppose that

$$w(s) \not\equiv w_0 \ (s \in \mathbb{R}), \tag{2.8}$$

$$w(s) \to w_0 \text{ as } s \to \infty \ \ (s \to -\infty) \tag{2.9}$$

and there exist $s_0 \in \mathbb{R}$ and $M > 0$ such that

$$s \geq s_0 \ \ (s \leq s_0) \Rightarrow w(s) \leq w_0 \ \ (w(s) \geq w_0) \text{ and } \|w'(s)\| \leq M. \tag{2.10}$$

Then there exist $\lambda < 0$ $(\lambda > 0)$ and a vector $q \in \mathbb{R}^N$ such that $q > 0$, and

$$(\lambda^2 A + \lambda cI + df[w_0])q = 0; \tag{2.11}$$

that is, there is a stable (unstable) monotone eigenvalue of the travelling-wave problem linearized at w_0.

Proof. We treat the case when $s \to \infty$; the other is similar. Without loss of generality, suppose $w_0 = 0$.

Consider first $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ that is bounded for s sufficiently large, satisfies (2.8) and (2.10), and

$$Aw''(s) + cw'(s) + Bw(s) = 0 \tag{2.12}$$

where $B \in P^{N\times N}$ and $\mu_{PF}(B) \neq 0$. Recall that the second order system (2.12) is equivalent to the first order system

$$\begin{pmatrix} w' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A^{-1}B & -cA^{-1} \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} =: K_c \begin{pmatrix} w \\ y \end{pmatrix} \tag{2.13}$$

where $y(s) := w'(s)$, $s \in \mathbb{R}$. Furthermore, the general solution of (2.13) is a linear combination of terms of the form

$$s^m e^{\beta s} p \text{ or } s^n e^{\gamma s} t(s),\ s \in \mathbb{R}, \tag{2.14}$$

where m, n are non-negative integers, β is a real eigenvalue of K_c with generalised eigenvector p, γ is the real part of a complex eigenvalue of K_c and $t(s)$ is a vector-valued combination of sines and cosines. It thus follows from the assumption that w is bounded for s sufficiently large that w has the form

$$w(s) = \left(\Sigma_{k=0}^m s^k q_k + \Sigma_{j=0}^n s^j \phi_j(s)\right) e^{\alpha s} + z(s). \tag{2.15}$$

Here $m, n \in \mathbb{N}$, $q_k \in \mathbb{R}^N$ for each $k \in \{0, \dots, m\}$ and for each $j \in \{0, \dots, n\}$, $\phi_j(s)$ is a vector-valued combination of $\cos \beta_j s$ and $\sin \beta_j s$, where β_j are non-zero imaginary parts of eigenvalues of K_c which have real part α; $\alpha \leq 0$ and the vector q_m (that multiplies the highest power of s) satisfies

$$(\alpha^2 A + \alpha c I + B) q_m = 0. \tag{2.16}$$

The function z comprises terms with a factor $e^{\beta s}$, where $\beta < \alpha$. Without loss of generality, m and n are chosen so that $q_m \neq 0$ unless $\Sigma_{k=0}^m s^k q_k \equiv 0$ and $\phi_n(s) \not\equiv 0$ unless $\Sigma_{j=0}^n s^j \phi_j(s) \equiv 0$. Also, $\Sigma_{k=0}^m s^k q_k + \Sigma_{j=0}^n s^j \phi_j(s) \not\equiv 0$ by condition (2.8).

Suppose first that $\Sigma_{k=0}^m s^k q_k \equiv 0$. Then $\phi_n(s) \not\equiv 0$ since $\Sigma_{k=0}^m s^k q_k + \Sigma_{j=0}^n s^j \phi_j(s) \not\equiv 0$ and

$$s^{-n} e^{-\alpha s} w(s) = \phi_n(s) + \Sigma_{j=0}^{n-1} s^{j-n} \phi_j(s) + s^{-n} e^{-\alpha s} z(s),\ s \in \mathbb{R}, \tag{2.17}$$

the second term on the right-hand side being included only if $n \geq 1$.

Bohr [4] defines an *almost-periodic* function to be a function f, continuous on $\mathbb{R}$, such that for each $\epsilon > 0$, there exists $L(\epsilon) > 0$ so that each interval of $\mathbb{R}$ of length $L(\epsilon)$ contains a number τ, satisfying

$$|f(x+\tau) - f(x)| \leq \epsilon \text{ for each } x \in \mathbb{R}. \tag{2.18}$$

It is shown in [4] that a finite sum of almost-periodic functions is almost-periodic. Since continuous periodic functions are clearly almost-periodic, it follows that ϕ_n is almost-periodic. Moreover, the mean value of ϕ_n, given by

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi_n(s)ds, \tag{2.19}$$

is zero, since for $\beta \in \mathbb{R}$,

$$|\int_0^T \cos\beta s ds| \leq \frac{1}{\beta} \tag{2.20}$$

and so $\int_0^T \phi_n(s)ds$ is bounded independently of $T > 0$. It is proved in [4] that a non-positive almost-periodic function with mean-value zero must be identically zero. Hence there exists a component $(\phi_n)_i$ of ϕ_n and $s_0 \in \mathbb{R}$ such that $(\phi_n)_i(s_0) > 0$. It follows from the definition of almost-periodicity above that there exists $\zeta > 0$ and a sequence $s_k \to \infty$ such that $(\phi_n)_i(s_k) > \zeta$. Since the second and third terms on the right-hand side of (2.17) tend to zero as $s \to \infty$, $w_i(s_k) > 0$ for k sufficiently large. But this contradicts the fact that $w(s) \leq 0$ for s sufficiently large (condition (2.10)). So $\Sigma_{k=0}^m s^k q_k \not\equiv 0$ (and hence $q_m \neq 0$, by the choice of m). Note also that $m \geq n$, since otherwise

$$s^{-n}e^{-\alpha s}w(s) = \phi_n(s) + \Sigma_{k=0}^m s^{k-n}q_k + \Sigma_{j=0}^{n-1} s^{j-n}\phi_j(s) + s^{-n}e^{-\alpha s}z(s) \tag{2.21}$$

and a similar argument to that in the last paragraph leads to a contradiction.

Suppose now that q_m has a positive component, say $q_{m_1} > 0$. (2.15) gives that

$$s^{-m}e^{-\alpha s}w(s) = q_m + \Sigma_{k=0}^{m-1} s^{k-m}q_k + \Sigma_{j=0}^{n} s^{j-m}\phi_j(s) + s^{-m}e^{-\alpha s}z(s), \ s \in \mathbb{R}. \tag{2.22}$$

(The second term on the right-hand side is included only if $m \geq 1$.) Recalling the above analysis for ϕ in the case when $m = n$, it follows that $w_1(s)$ takes positive values on a sequence tending to infinity. This contradicts that $w(s) \leq 0$ for s sufficiently large, as before.

Hence $q_m \leq 0$. Since $q_m \neq 0$, it follows from (2.16) and the facts that A is diagonal and $B \in P^{N\times N}$ that $q_m < 0$. If $\alpha = 0$, (2.16) yields that $\mu_{PF}(B) = 0$, since $-q_m > 0$, and any *positive* eigenvector corresponds to the Perron-Frobenius eigenvalue (see Theorem 2.2.1). This contradicts the hypotheses on B, so $\alpha < 0$. Thus the conclusion of the theorem holds for the linear problem (2.12), taking $q = -q_m$.

Return now to w satisfying the hypotheses of the theorem. If $w(s) = 0$ for s sufficiently large, then $w(s) \equiv 0$ by the uniqueness of solutions to initial value problems. But $w(s) \not\equiv 0$ by (2.8). So since $w(s) \to 0$ as $s \to \infty$, there is a sequence $s_n \to \infty$ such that for n sufficiently large,

$$\frac{1}{n} = \|w(s_n)\| = \sup_{s \geq s_n} \|w(s)\|. \tag{2.23}$$

Define functions $w_n : \mathbb{R} \to \mathbb{R}^N$ by $w_n(s) = w(s+s_n)$; in particular, $w_n(0) = w(s_n)$. Also, write $f(y) = By + R(y)$, $y \in \mathbb{R}^N$, where $B := df[0]$ and $R : \mathbb{R}^N \to \mathbb{R}^N$ is such that $\dfrac{\|R(y)\|}{\|y\|} \to 0$ as $\|y\| \to 0$. (2.7) becomes

$$Aw_n''(s)+cw_n'(s)+\sigma(s+s_n)G(w_n(s),w_n'(s))w_n'(s)+Bw_n(s)+R(w_n(s)) = 0, \;\; s \in \mathbb{R}. \tag{2.24}$$

Now defining $v_n(s) = \dfrac{w_n(s)}{\|w_n(0)\|}$ for $s \geq 0$ yields that

$$Av_n''(s)+cv_n'(s)+\sigma(s+s_n)G(w_n(s),w_n'(s))v_n'(s)+Bv_n(s)+\frac{R(w_n(s))}{\|w_n(0)\|} = 0, \;\; s \geq 0. \tag{2.25}$$

We require that $\dfrac{\|R(w_n(s))\|}{\|w_n(0)\|} \to 0$ uniformly on $[0,\infty)$ as $n \to \infty$. Let $\epsilon > 0$ be given. There exists $t \in \mathbb{R}$ such that

$$s \geq t \Rightarrow \|R(w(s))\| \leq \epsilon \|w(s)\|. \tag{2.26}$$

Also, $s_n \to \infty$ as $n \to \infty$, so there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow s + s_n \geq t \text{ for every } s \geq 0. \tag{2.27}$$

Hence for $n \geq n_0$,

$$\frac{||R(w_n(s))||}{||w_n(0)||} \leq \frac{||R(w_n(s))||}{||w_n(s)||} = \frac{||R(w(s+s_n))||}{||w(s+s_n)||} \leq \epsilon \text{ for every } s \geq 0, \tag{2.28}$$

from which the required convergence follows.

The aim now is to find a subsequential limit of $\{v_n\}$, using the Arzela-Ascoli theorem, that is bounded for s sufficiently large, and satisfies (2.8), (2.10) and the linear system (2.12), with $B = df[0]$. The result will then follow from the above analysis for the linear system.

For this, bounds on $v_n'(s)$ and $v_n''(s)$ are required. Note first that by (2.7), (2.9) and (2.10), there exists $t_0 \in \mathbb{R}$ such that $||w(s)||$, $||w'(s)||$ and $||w''(s)||$ are bounded independently of $s \geq t_0$. Hence for $n \geq n_1$, say, $||w_n(s)||$, $||w_n'(s)||$ and $||w_n''(s)||$ are bounded independently of $n \geq n_1$ and $s \geq 0$, and for each *fixed* $n \geq n_1$, $||v_n(s)||$, $||v_n'(s)||$ and $||v_n''(s)||$ are bounded independently of $s \geq 0$. Also, since $||w_n||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}$ and $||w_n'||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}$ are bounded independently of $n \geq n_1$ and G is continuous, there exists $M_1 > 0$ such that

$$||G(w_n(s), w_n'(s))|| \leq M_1 \text{ for } n \geq n_1, \; s \geq 0. \tag{2.29}$$

Now Landau's inequality [1] together with (2.25) yield that

$$\begin{aligned}
&||v_n'||^2_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq 4||v_n||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}||v_n''||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \\
&= \; 4||v_n||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \left\| A^{-1}(cv_n' + \sigma G(w_n, w_n')v_n' + Bv_n + R(w_n)||w_n(0)||^{-1}) \right\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \\
&\leq \; 4||A^{-1}|| \left\{ |c| ||v_n'||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} + M_1 ||v_n'||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \right. \\
&\qquad \left. + ||B|| + ||R(w_n)||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}||w_n(0)||^{-1} \right\}
\end{aligned} \tag{2.31}$$

since $||v_n(s)|| = \frac{||w_n(s)||}{||w_n(0)||} \leq 1$ for each n, $s \geq 0$. Since $||R(w_n(s))|| ||w_n(0)||^{-1} \to 0$ as $n \to \infty$ *uniformly* for $s \geq 0$, $||R(w_n)||_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}||w_n(0)||^{-1}$ is bounded indepen-

[1] A scalar-valued version of Landau's inequality is proved in Hille [23]. The following vector-valued version is a straightforward consequence.

Landau's Inequality

Let $\Lambda \subset \mathbb{R}$ be either a semi-infinite interval or $\mathbb{R}$. Let $v \in C^2(\Lambda, \mathbb{R}^N)$ be such that v, v' and v'' are uniformly bounded on Λ. Then

$$||v'||^2_{L_\infty(\Lambda,\mathbb{R}^N)} \leq 4||v||_{L_\infty(\Lambda,\mathbb{R}^N)}||v''||_{L_\infty(\Lambda,\mathbb{R}^N)}. \tag{2.30}$$

dently of n, say by $M_2 > 0$. Hence for $n \geq n_1$,

$$\|v_n'\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq \max\left\{1, 4\|A^{-1}\|(|c| + M_1 + M_2 + \|B\|)\right\} =: N_1. \tag{2.32}$$

It then follows immediately from (2.32) and (2.25) that $\|v_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}$ is bounded independently of $n \geq n_1$, say by $N_2 > 0$.

Now let $r > 0$ and define $X_r := [0, r]$. Then $\|v_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq 1$ for all n, so the sequence $\{v_n\}_{n=n_1}^\infty$ is uniformly bounded on X_r. Moreover, the mean value inequality for vector-valued functions gives that for $n \geq n_1$, $s, t \in X_r$, there exists $\theta \in (0, 1)$ such that for $\xi = \theta s + (1 - \theta)t$,

$$\|v_n(s) - v_n(t)\| \leq \|v_n'(\xi)\||s - t| \leq N_1|s - t|, \tag{2.33}$$

where N_1 is defined in (2.32). Hence $\{v_n\}_{n=n_1}^\infty$ is equicontinuous. The Arzela-Ascoli Theorem thus yields that there is a subsequence v_k and $u_1 \in C(X_r, \mathbb{R}^N)$ such that $v_k \to u_1$ in $C(X_r, \mathbb{R}^N)$. Since $\|v_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} < N_2$ for $n \geq n_1$, there is a further subsequence $\{v_j\}$ of $\{v_k\}$, and $u_2 \in C(X_r, \mathbb{R}^N)$, such that $v_j' \to u_2$ in $C(X_r, \mathbb{R}^N)$.

Now for $n \geq n_1$, $\|w_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} = \|w_n(0)\| = \dfrac{1}{n} \to 0$ as $n \to \infty$, and by Landau's inequality,

$$\|w_n'\|^2_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq 4\|w_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}\|w_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq \frac{4\gamma}{n} \tag{2.34}$$

since there exists $\gamma > 0$ such that $\|w_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq \gamma$ for each $n \geq n_1$. So since $G(w_0, 0) = 0$ for each $w_0 \in \mathbb{R}^N$ for which $f(w_0) = 0$ (condition **(G3)**) and G is continuous,

$$G(w_n(s), w_n'(s)) \to 0 \text{ as } n \to \infty, \tag{2.35}$$

uniformly for $s \in X_r$. The fact that $\|v_m'(s)\| \leq N_1$ for $s \geq 0$, $m \geq n_1$ thus yields that

$$G(w_n(s), w_n'(s))v_n'(s) \to 0 \text{ as } n \to \infty, \tag{2.36}$$

uniformly for $s \in X_r$.

It thus follows from (2.25), (2.36), the convergence of v_j, v_j' and the fact that $\sigma(s) \in [0, 1]$ for each $s \in \mathbb{R}$ that the sequence $\{v_j''\}$ is uniformly convergent on

X_r. Hence $\{v_j\}$ is Cauchy in the Banach space $C^2(X_r, \mathbb{R}^N)$ (see section 2.1 for a precise definition) and thus there exists $v \in C^2(X_r, \mathbb{R}^N)$ such that $v_j \to v$ in $C^2(X_r, \mathbb{R}^N)$ as $j \to \infty$. So for each $r > 0$, there is a subsequence of the original sequence $\{v_n\}$ that converges in $C^2(X_r, \mathbb{R}^N)$; a diagonal subsequence argument now shows that $\{v_n\}$ has a subsequence that converges in $C^2([0,r], \mathbb{R}^N)$ for each $r > 0$, to a limit $v \in C^2(\mathbb{R}_+, \mathbb{R}^N)$, the space of $\mathbb{R}^N$-valued functions that are twice continuously differentiable on $(0, \infty)$ and continuous on $\mathbb{R}_+$. Letting $n \to \infty$ in (2.25), taking (2.26) and (2.36) into account, yields that

$$Av''(s) + cv'(s) + Bv(s) = 0 \text{ for each } s > 0. \tag{2.37}$$

By construction of v, it is clear that $\|v(s)\|$ and $\|v'(s)\|$ are both uniformly bounded for $s > 0$, and that $v(s) \leq 0$ for each $s > 0$. Moreover, $\|v_n(0)\| = \dfrac{\|w_n(0)\|}{\|w_n(0)\|} = 1$ for each $n \in \mathbb{N}$, so $\|v(0)\| = 1$ since $v_j \to v$ in $C^2([0,1])$ (for example). So $v(0) \neq 0$. Thus v satisfies the hypotheses for the linear analysis above. The result follows.

□

Chapter 3

Eigenvalues of the travelling-wave problem

Here we turn to eigenvalues of the travelling-wave problem, and their dependence on the wave velocity c. In each of the bistable and monostable cases, two results are elucidated, one on the existence of monotone eigenvalues, and the other on the dimensions on the stable and unstable manifolds of the travelling-wave problem. (Recall the definitions of these concepts from Chapter 1.)

We begin with some observations on the continuity of eigenvalues. Let $A \in M^{N\times N}$ be a positive-definite diagonal matrix, and let $B \in P^{N\times N}$. The matrix $M(\lambda, c) := \lambda^2 A + \lambda c I + B$ depends analytically on λ and on c. Thus the eigenvalues of $M(\lambda, c)$ (that is, $\gamma \in \mathbb{C}$ with corresponding $z \in \mathbb{C}^N \backslash \{0\}$ such that $M(\lambda, c)z = \gamma z$, as distinct from the travelling-wave eigenvalues λ for a given c, which satisfy $\det(M(\lambda, c)) = 0$) are clearly bounded for (λ, c) in a bounded set in $\mathbb{C}\times\mathbb{R}$ and they vary continuously with λ and with c. (See Kato [26], pages 63-64.) Moreover, for each fixed λ and c, $M(\lambda, c) \in P^{N\times N}$ has a Perron-Frobenius eigenvalue, $\mu_{PF}(M(\lambda, c))$, which is real and *simple*. The Implicit Function Theorem for real-analytic functions (see Dieudonné [15], page 272) thus gives that $\mu_{PF}(M(\lambda, c))$ is a real analytic function of λ for each fixed c. Clearly, $\mu_{PF}(M(\lambda, c))$ is an affine function of c for fixed λ.

Note also that the eigenvalues λ of the travelling-wave problem cannot escape to infinity at a finite value of c, as the next lemma shows.

Lemma 3.0.2 *Let $A \in M^{N\times N}$ be a non-singular diagonal matrix, $B \in M^{N\times N}$*

and $R > 0$. Then for $c \in \mathbb{R}$, $|c| \leq R$, the solutions λ of $\det(\lambda^2 A + \lambda c I + B) = 0$ *form a bounded set in $\mathbb{C}$.*

Proof. Suppose, for contradiction, that for all $k \in \mathbb{N}$, there exists $c_k \in \mathbb{R}$ with $|c_k| \leq R$, $|\lambda_k| > k$ and $z^k \in \mathbb{C}^N \backslash \{0\}$ such that

$$(\lambda_k^2 A + \lambda_k c_k I + B) z^k = 0. \tag{3.1}$$

Since $z^k \neq 0$, we can assume that $\| z^k \|_{\mathbb{C}^N} = 1$. Now $\{c_k\}$, a bounded sequence in $\mathbb{R}$, has a convergent subsequence, say $c_j \to c$. Also, the corresponding subsequence $\{z^j\}$ is bounded in $\mathbb{C}^N$, so there is a further subsequence such that $c_m \to c$, $z^m \to z$, and $\| z \|_{\mathbb{C}^N} = 1$ since $\| z^m \|_{\mathbb{C}^N} = 1$ for each m. Since $\lambda_m \neq 0$ for each m,

$$(\lambda_m A + c_m I + \lambda_m^{-1} B) z^m = 0 \tag{3.2}$$

and thus because $\lambda_m^{-1} \to 0$ as $m \to \infty$, $\lambda_m A z^m \to -cz$. As A is diagonal, $\lambda_m A_{ii} z_i^m \to -c z_i$ for each i, $1 \leq i \leq N$. Thus since $|\lambda_m| \to \infty$ and A is non-singular, we must have $z_i^m \to 0$. This contradicts that $z^m \to z$ with $\| z \|_{\mathbb{C}^N} = 1$.

□

3.1 The bistable case

Consider first the travelling-wave problem described in Chapter 1 with the additional condition that both S and T are stable equilibria of f. The continuity observations above allow proof of the following.

Theorem 3.1.1 *Let $B \in P^{N \times N}$ be such that $\mu_{PF}(B) < 0$. Then for each $c \in \mathbb{R}$, there is a stable and an unstable monotone eigenvalue of the travelling-wave problem (1.3) with linearization B.*

Proof. Fix $c \in \mathbb{R}$. Observe first that $\mu_{PF}(M(0,c)) = \mu_{PF}(B) < 0$. Next, if $q := (1, \ldots, 1) \in \mathbb{R}^N$,

$$(A + \lambda^{-1} c I + \lambda^{-2} B) q > 0 \tag{3.3}$$

for $\lambda \in \mathbb{R}$ with $|\lambda|$ sufficiently large since A is a positive-definite diagonal matrix. Thus $M(\lambda, c) q > 0$ and by Corollary 2.2.4, $\mu_{PF}(M(\lambda, c)) > 0$. Now $\mu_{PF}(M(\cdot, c))$

is a continuous real valued function of λ, which is negative when $\lambda = 0$ and positive for $|\lambda|$ large. So the Intermediate Value Theorem gives that there exist $\lambda_1 < 0$ and $\lambda_2 > 0$ with

$$\mu_{PF}(M(\lambda_1, c)) = \mu_{PF}(M(\lambda_2, c)) = 0. \tag{3.4}$$

That the corresponding eigenvectors are positive follows immediately from the Perron-Frobenius Theorem.

□

This shows that for all $c \in \mathbb{R}$, there is an unstable monotone direction at S and a stable monotone direction at T. Now we study the dimensions of the stable and unstable manifolds at a stable equilibrium point. In the bistable case, it is straightforward to show that $D_s(S) + D_u(T) \geq 2N$, using the fact that there are no eigenvalues of the travelling-wave problem, linearized at a stable equilibrium, on the imaginary axis.

Theorem 3.1.2 *Let $B \in P^{N\times N}$ be such that $\mu_{PF}(B) < 0$. Then for each $c \in \mathbb{R}$, there are N eigenvalues of the travelling-wave problem in the open left- and right-half planes.*

Proof. Observe first that since A is positive-definite, $\det(\lambda^2 A + \lambda c I + B)$ is a polynomial in λ of degree $2N$ for any fixed $c \in \mathbb{R}$. So counting algebraic multiplicities, there are $2N$ solutions λ of $\det(\lambda^2 A + \lambda c I + B) = 0$ in $\mathbb{C}$.

Now zero is not an eigenvalue of the travelling-wave problem for any value of c since $\mu_{PF}(B) < 0$. Suppose that for some $c \in \mathbb{R}$, and $\alpha \in \mathbb{R}\backslash\{0\}, i\ \alpha$ is an eigenvalue of the travelling-wave problem. Then there exists $z \in \mathbb{C}^N\backslash\{0\}$ such that $(-\alpha^2 A + i\alpha c I + B)z = 0$ and $-i\alpha c$ is an eigenvalue of $B - \alpha^2 A \in P^{N\times N}$. Hence $\mu_{PF}(B - \alpha^2 A) \geq 0$ and there exists $x > 0$ such that $(B - \alpha^2 A)x = (\mu_{PF}(B - \alpha^2 A))x$. Therefore $Bx = (\mu_{PF}(B - \alpha^2 A))x + \alpha^2 A x > 0$. Corollary 2.2.4 then yields that $\mu_{PF}(B) > 0$, which is false. Thus for each c, there are no eigenvalues of the travelling-wave problem with linearization at B on the imaginary axis.

When $c = 0$, travelling-wave eigenvalues λ satisfy $\det(\lambda^2 A + B) = 0$. Then λ is a solution whenever $-\lambda$ is and the algebraic multiplicity of a solution λ is equal to that of $-\lambda$. It is known that there can be no eigenvalues on the imaginary axis, and hence counting algebraic multiplicities, there must be N in each open half plane since there are $2N$ solutions λ of $\det(\lambda^2 A + B) = 0$. All that remains is to observe that Lemma 3.0.2 implies that the eigenvalues vary continuously with c, and they do not cross the imaginary axis by the first part of this proof. Hence for each $c \in \mathbb{R}$, there are N eigenvalues in each open half plane.

□

The dimensions of the unstable and stable manifolds at S and T for all values of c are immediate:

$$D_u(S) = D_u(T) = D_s(S) = D_s(T) = N.$$

3.2 The monostable case

Throughout this section, S is a stable equilibrium ($\mu_{PF}(df[S]) < 0$) and T is unstable ($\mu_{PF}(df[T]) > 0$). First note that Theorems 3.1.1 and 3.1.2 yield that, for each c, there is an unstable monotone eigenvalue of the travelling-wave problem linearized at S and that the unstable manifold at S has dimension N. We start by discussing the existence and behaviour of stable monotone eigenvalues at an unstable equilibrium.

Lemma 3.2.1 *Let $B \in P^{N\times N}$ with $\mu_{PF}(B) > 0$. Then*

(i) *For $c \leq 0$, there are no stable monotone eigenvalues of the travelling-wave problem with linearization B.*

(ii) *For $c > 0$ sufficiently large, there exists a stable monotone eigenvalue of the travelling-wave problem with linearization B.*

(iii) *If there exists a stable monotone eigenvalue for $c = c^* > 0$, then for all $c \geq c^*$, a stable monotone eigenvalue exists.*

Proof. Let $x > 0$ be the Perron-Frobenius eigenvector of B and recall that $M(\lambda, c) := \lambda^2 A + \lambda c I + B$.

(i) Take $\lambda < 0$ and $c \le 0$. Then $\lambda^2 A + B \in P^{N\times N}$ and $(\lambda^2 A + B)x = \lambda^2 Ax + \mu_{PF}(B)x > 0$, so by Corollary 2.2.4, $\mu_{PF}(\lambda^2 A + B) > 0$. Let $y > 0$ be the Perron-Frobenius eigenvector of $\lambda^2 A + B$. Then $(\lambda^2 A + B)y = \mu_{PF}(\lambda^2 A + B)y$ so that $(\lambda^2 A + \lambda cI + B)y = (\mu_{PF}(\lambda^2 A + B) + \lambda c)y$. Since an eigenvector which is positive is a Perron-Frobenius eigenvector, $\mu_{PF}(M(\lambda, c)) = \lambda c + \mu_{PF}(M(\lambda, 0)) > 0$. Hence given $c \le 0$, $\mu_{PF}(M(\lambda, c)) > 0$ for $\lambda < 0$ and thus there is no stable monotone eigenvalue.

(ii) $(A+B)x = Ax + \mu_{PF}(B)x > 0$. Hence by Corollary 2.2.4, $\mu_{PF}(A+B) > 0$, and so for some $y > 0$, $(A + B)y = \mu_{PF}(A + B)y$. Then $(A - cI + B)y = (\mu_{PF}(A+B) - c)y$ and thus $\mu_{PF}(M(-1, c)) = \mu_{PF}(A - cI + B) = \mu_{PF}(A+B) - c$. So $\mu_{PF}(M(-1, c)) < 0$ for $c > \mu_{PF}(A + B)$. But $\mu_{PF}(M(0, c)) = \mu_{PF}(B) > 0$. Hence, by the continuous dependence of the eigenvalues of $M(\lambda, c)$ on λ, there exists λ_0 with $-1 < \lambda_0 < 0$ and $\mu_{PF}(M(\lambda_0, c)) = 0$. That is, λ_0 is a stable monotone eigenvalue.

(iii) Suppose that $\lambda < 0$ is a stable monotone eigenvalue at $c = c^*$ with corresponding eigenvector $y > 0$ so that $(\lambda^2 A + \lambda c^* I + B)y = 0$. Then for $c > c^*$, $(A\lambda^2 + \lambda cI + B)y = \lambda(c - c^*)y$. So $\mu_{PF}(M(\lambda, c)) = \lambda(c - c^*) < 0$. Again, by continuity of the dependence of $\mu_{PF}(M(\lambda, c))$ on λ, there exists λ_0 with $\lambda < \lambda_0 < 0$, and $\mu_{PF}(M(\lambda_0, c)) = 0$, since $\mu_{PF}(M(0, c)) > 0$. The result follows.

$\square$

So for c sufficiently large and positive, a stable monotone eigenvalue of the travelling-wave problem linearized at T exists. In fact, the set of c for which such an eigenvalue exists is $[c_0, \infty)$ for some $c_0 > 0$.

Theorem 3.2.2 *Let $B \in P^{N\times N}$ with $\mu_{PF}(B) > 0$. Then there exists $c_0 > 0$ such that for each $c \ge c_0$, there is a stable monotone eigenvalue of the travelling-wave problem with linearization B, and for $c < c_0$, there is no such λ.*

Proof. It follows immediately from Lemma 3.2.1 (i),(ii) that the set $V := \{c \in \mathbb{R} : (A\lambda^2 + \lambda cI + B)y = 0$, for some $\lambda < 0, y > 0\}$ is non-empty and bounded below by 0, so has a non-negative infimum, say c_0. Consider sequences $c_k \to c_0$, with corresponding $\lambda_k < 0$ and $z^k > 0$ such that

$$(\lambda_k^2 A + \lambda_k c_k I + B)z^k = 0, \; \| z^k \| = 1 \tag{3.5}$$

Taking a subsequence if necessary, we can assume that $z^k \to z$ where $\| z \|= 1$, and thus, since $\lambda_k \neq 0$ and $\mu_{PF}(B) > 0$, there results that $(\lambda_k A + \lambda_k^{-1} B) z^k \to -c_0 z$ as $k \to \infty$. So, by the argument given in Lemma 3.0.2, $\{\lambda_k\}$ is bounded, and hence has a convergent subsequence, $\lambda_n \to \lambda \leq 0$. Passing to the limit in (3.5) gives that $(\lambda^2 A + \lambda c_0 I + B)z = 0$. It is immediate that $z \geq 0$; in fact, $z > 0$ because the off-diagonal elements of B are strictly positive. Now it is clear that $\lambda < 0$ since otherwise $\lambda = 0$, which contradicts $\mu_{PF}(B) > 0$. Therefore $c_0 \in V$ as required. That V contains all $c \geq c_0$ is the result of Lemma 3.2.1 (iii).

□

Even more is true: for $c > c_0$, there are precisely two stable monotone eigenvalues, both of which depend monotonically on c. Although not directly related to the existence of monotone eigenvalues, this analysis eventually yields information on the dimension of the stable manifold at T. The first step is to recall a convexity result due to Cohen [10].

Theorem 3.2.3 *Let $M \in P^{N\times N}$ and $D \in M^{N\times N}$ be diagonal. Then the Perron-Frobenius eigenvalue of $M + D$ is a convex function of D; that is, given diagonal matrices D_1 and D_2 and $0 < \alpha < 1$,*

$$\mu_{PF}(\alpha D_1 + (1-\alpha)D_2 + M) \leq \alpha \mu_{PF}(D_1 + M) + (1-\alpha)\mu_{PF}(D_2 + M).$$

Particularly elementary proofs of this result are given in [17] and [36], the second of which unifies Theorem 3.2.3 with a log-convexity result due to Kingman [27]. For the travelling-wave eigenvalue problem, Theorem 3.2.3 has the following elegant consequence.

Lemma 3.2.4 *Let $B \in P^{N\times N}$, $c \in \mathbb{R}$ and let A be a positive-definite diagonal matrix. Then the Perron-Frobenius eigenvalue $\mu_{PF}(\lambda^2 A + \lambda c I + B)$ is a strictly convex function of λ.*

Proof. Recall first that increasing any element of a matrix with positive off-diagonal elements increases its Perron-Frobenius eigenvalue (Theorem 2.2.2).

Now for $\lambda_1, \lambda_2 \in \mathbb{R}$ ($\lambda_1 \neq \lambda_2$) and $0 < t < 1$, $(t\lambda_1 + (1-t)\lambda_2)^2 < t\lambda_1^2 + (1-t)\lambda_2^2$. So since A is positive-definite,

$$\begin{aligned} & \mu_{PF}((t\lambda_1 + (1-t)\lambda_2)^2 A + (t\lambda_1 + (1-t)\lambda_2)cI + B) \\ < \; & \mu_{PF}((t\lambda_1^2 + (1-t)\lambda_2^2)A + (t\lambda_1 + (1-t)\lambda_2)cI + B) \\ = \; & \mu_{PF}(t(\lambda_1^2 A + \lambda_1 cI) + (1-t)(\lambda_2^2 A + \lambda_2 cI) + B) \\ \leq \; & t\mu_{PF}(\lambda_1^2 A + \lambda_1 cI + B) + (1-t)\mu_{PF}(\lambda_2^2 A + \lambda_2 c + B) \end{aligned}$$

Since $\lambda_k^2 A + \lambda_k cI$ are diagonal ($k = 1, 2$), Theorem 3.2.3 gives the last inequality.

□

Now we can determine the exact number of stable monotone eigenvalues for a given velocity c.

Lemma 3.2.5 *Let $B \in P^{N \times N}$ with $\mu_{PF}(B) > 0$ and let c_0 be the critical velocity in Theorem 3.2.2. Then at $c = c_0$, there is exactly one stable monotone eigenvalue, and for $c > c_0$, there are precisely two.*

Proof. Fix $c > c_0$. As in Lemma 3.2.1 (iii), there exists $\lambda_0 < 0$ with $\mu_{PF}(M(\lambda_0, c)) < 0$. Now $\mu_{PF}(B) > 0$ and by the argument in Theorem 3.1.1, $\mu_{PF}(M(\lambda, c)) > 0$ for $|\lambda|$ large. For a given c, $\mu_{PF}(M(\cdot, c))$ is a continuous function of λ, so there exist $\lambda_1 < \lambda_0 < \lambda_2 < 0$ such that $\mu_{PF}(M(\lambda_1, c)) = \mu_{PF}(M(\lambda_2, c)) = 0$. Since $\mu_{PF}(M(\cdot, c))$ is a strictly convex function of λ, there can be at most two real solutions λ of $\mu_{PF}(M(\lambda, c)) = 0$, and hence there are precisely two. For $c = c_0$ there is at least one stable monotone eigenvalue and at most two by convexity. If there were two, say $\lambda_1 < \lambda_2$, then for fixed λ with $\lambda_1 < \lambda < \lambda_2$, $\mu_{PF}(M(\lambda, c_0)) < 0$. Hence for $c < c_0$ sufficiently close to c_0, $\mu_{PF}(M(\lambda, c)) < 0$. But as $\mu_{PF}(M(0, c)) = \mu_{PF}(B) > 0$, this implies that there exists λ_0 with $\lambda < \lambda_0 < 0$ and $\mu_{PF}(M(\lambda_0, c)) = 0$, which contradicts the minimality of c_0. Hence it follows that there is exactly one stable monotone eigenvalue at c_0.

□

To determine the dimension of the stable manifold at T from the behaviour of the monotone eigenvalues, we need to relate the other eigenvalues of the travelling-wave problem to the stable monotone ones. Vol'pert and Vol'pert [44] have shown the following useful result, the proof of which is included for completeness.

Lemma 3.2.6 *Let $M \in P^{N\times N}$ with $\mu_{PF}(M) < 0$ and S be a complex diagonal $N \times N$ matrix. If the diagonal elements of S have non-positive real parts then all eigenvalues of the matrix $M + S$ have negative real parts.*

Proof. Let $d > 0$ be the Perron-Frobenius eigenvector of M and D be a diagonal matrix with the elements of the vector $d = (d_1, \ldots, d_N)$ on the diagonal. Then the sum of the elements of the rows of the matrix $D^{-1}MD$ is negative. Now the matrices $M + S$ and $D^{-1}(M + S)D = D^{-1}MD + S$ have the same eigenvalues. By Gerschgorin's theorem (see, for example, [46], page 71), the eigenvalues of the matrix $D^{-1}MD + S$ lie in disks with centres at the points $M_{jj} + S_{jj}$ and radii $\sum_{k,k\neq j} d_j^{-1} M_{jk} d_k$. Since $\sum_k d_j^{-1} M_{jk} d_k + \operatorname{Re} S_{jj} < 0$, all the Gerschgorin disks are in the left-half plane.

□

Theorem 3.2.7 *Let B and c_0 be as in Lemma 3.2.5 and denote the two stable monotone eigenvalues at $c > c_0$ by λ_1 and λ_2, where $\lambda_1 < \lambda_2 < 0$. Then there are no eigenvalues of the travelling-wave problem λ with $\lambda_1 < Re\ \lambda < \lambda_2$.*

Proof. For $\gamma \in \mathbb{R}$ with $\lambda_1 < \gamma < \lambda_2$, $\mu_{PF}(M(\gamma, c)) < 0$ by the convex dependence of $\mu_{PF}(M(\lambda, c))$ on λ. Suppose that $\gamma + i\beta$ is an eigenvalue of the travelling-wave problem for some $\beta \in \mathbb{R}$. Then there exists $z \in \mathbb{C}^N \backslash \{0\}$ such that

$$((\gamma + i\beta)^2 A + (\gamma + i\beta)cI + B)z = 0.$$

But then zero is an eigenvalue of the matrix $(M(\gamma, c) - \beta^2 A + i\beta(2\lambda A + cI))$, which contradicts Lemma 3.2.6.

□

Remark. Suppose that zero is an eigenvalue of the matrix $M(\lambda, c)$ for fixed λ and c, in the sense that there exists $z \in \mathbb{C}^N \backslash \{0\}$ such that $M(\lambda, c)z = \gamma z$ where $\gamma = 0$. Then λ is an eigenvalue of the travelling-wave problem for velocity c, as defined in Chapter 1. Since the eigenspace of zero as an eigenvalue of the matrix $M(\lambda, c)$ is the same as the eigenspace of the travelling-wave eigenvalue λ for velocity c, the dimensions of these two spaces are the same. In other words, the *geometric* multiplicities of the two eigenvalues coincide. However, the *algebraic* multiplicity of zero as an eigenvalue of $M(\lambda, c)$ may not equal the algebraic multiplicity of λ as a solution of the equation $\det(\lambda^2 A + \lambda c I + B) = 0$. In particular, $\mu_{PF}(M(\lambda_0, c_0)) = 0$, where λ_0 is the unique stable monotone eigenvalue at $c = c_0$, and hence zero is a *simple* eigenvalue of $M(\lambda_0, c_0)$. Thus the geometric and algebraic multiplicities of zero as an eigenvalue of $M(\lambda_0, c_0)$ are both one. But the next result shows that two travelling-wave eigenvalues converge to λ_0 as c tends to c_0. So the algebraic multiplicity of λ_0 must be at least two.

Theorem 3.2.8 *Let B and c_0 be as in Lemma 3.2.5 and denote the two stable monotone eigenvalues for $c > c_0$ by $\lambda_1(c)$ and $\lambda_2(c)$ where $\lambda_1(c) < \lambda_2(c) < 0$. Let λ_0 denote the unique stable monotone eigenvalue when $c = c_0$. Then λ_1 is a continuous, decreasing function of c and λ_2 is a continuous, increasing function of c. Moreover, as $c \to \infty, \lambda_1(c) \to -\infty$ and $\lambda_2(c) \to 0$. As $c \to c_0$, $\lambda_1(c) \to \lambda_0$ and $\lambda_2(c) \to \lambda_0$ and as c is decreased below c_0, a pair of complex conjugate branches of eigenvalues emanate from λ_0.*

Proof. It is easy to show that $\lambda_1(c)$ and $\lambda_2(c)$ are widely separated for large c. For $\gamma < 0$ let $y > 0$ be such that

$$(\gamma^2 A + B)y = \mu_{PF}(M(\gamma, 0))y. \tag{3.6}$$

Then

$$(\gamma^2 A + \gamma c I + B)y = (\mu_{PF}(M(\gamma, 0)) + \gamma c)y, \tag{3.7}$$

so for c sufficiently large, $\mu_{PF}(M(\gamma, c)) < 0$. But $\mu_{PF}(M(0, c)) > 0$ for each c and it was shown in Theorem 3.1.1 that $\mu_{PF}(M(\lambda, c)) > 0$ for fixed c, $|\lambda|$ sufficiently large. So

$$\lambda_1(c) < \gamma < \lambda_2(c) \tag{3.8}$$

for c large enough. Since $\gamma < 0$ was arbitrary, $\lambda_1(c) \to -\infty$ and $\lambda_2(c) \to 0$ as $c \to \infty$.

Now let $c > c_0$. By the convex dependence of $\mu_{PF}(M(\lambda, c))$ on λ for fixed c, there exists ϵ_0 with $0 < \epsilon_0 < (\lambda_0 - \lambda_1(c))$ such that $\mu_{PF}(M(\lambda, c)) > 0$ for $\lambda \in (\lambda_1(c) - \epsilon_0, \lambda_1(c))$ and $\mu_{PF}(M(\lambda, c)) < 0$ for $\lambda \in (\lambda_1(c), \lambda_1(c) + \epsilon_0)$. By algebra, $\mu_{PF}(M(\lambda_1(c), c - \delta) = -\delta\lambda_1(c) > 0$ for all $\delta > 0$. Furthermore, for $\epsilon \in (0, \epsilon_0)$ and fixed $\delta > 0$ sufficiently small, $\mu_{PF}(M(\lambda_1(c) + \epsilon, c - \delta)) = \mu_{PF}(M(\lambda_1(c) + \epsilon, c) - \delta(\lambda_1(c) + \epsilon) < 0$. Thus by the continuous dependence of $\mu_{PF}(M(\lambda, c))$ on λ for fixed c, given $\epsilon \in (0, \epsilon_0)$, there exists $\delta_\epsilon > 0$ such that for each $\delta \in (0, \delta_\epsilon)$, there exists some $\tilde{\epsilon} \in (0, \epsilon)$ such that $\mu_{PF}(M(\lambda_1(c) + \tilde{\epsilon}, c - \delta)) = 0$. By choice of ϵ_0, $\lambda_1(c) + \tilde{\epsilon} < \lambda_0$, and hence $\lambda_1(c - \delta) = \lambda_1(c) + \tilde{\epsilon} > \lambda_1(c)$. Therefore, the stable monotone eigenvalue further from 0 increases as c decreases. Together with the analogous argument for $\lambda_1(c + \delta)$, $\delta > 0$ small, this also shows that λ_1 is a continuous function of c. Similarly, λ_2 is a continuous, increasing function of c.

To see that $\lambda_1(c) \to \lambda_0$ and $\lambda_2(c) \to \lambda_0$ as $c \to c_0$, first note that the proof of Lemma 3.2.5 gives that at $c = c_0$, $\mu_{PF}(M(\lambda, c_0)) > 0$ for $\lambda \in \mathbb{R}\backslash\{\lambda_0\}$ whilst $\mu_{PF}(M(\lambda_0, c_0)) = 0$. The preceding argument yields that there exists $\epsilon_0 > 0$ such that, given $\epsilon \in (0, \epsilon_0)$, there exists $\delta_\epsilon > 0$ such that for $\delta \in (0, \delta_\epsilon)$, $\mu_{PF}((M(\lambda_0, c_0 + \delta)) = \delta\lambda_0 < 0$, $\mu_{PF}(M(\lambda_0 + \epsilon, c_0 + \delta)) > 0$ and $\mu_{PF}(M(\lambda_0 - \epsilon, c_0 + \delta)) > 0$. Hence $\lambda_0 - \epsilon < \lambda_1(c_0 + \delta) < \lambda_0 < \lambda_2(c_0 + \delta) < \lambda_0 + \epsilon$.

The last step is to note that after 'coalescence' at $c = c_0$, the extensions of the continuous branches of stable monotone eigenvalues cannot remain real, and hence must become a complex conjugate pair. Suppose to the contrary. Now zero is the Perron-Frobenius eigenvalue of the matrix $\lambda_0^2 A + \lambda_0 c_0 I + B$, and is therefore simple – in particular, it has geometric multiplicity one. (See the remark preceding this theorem.) Therefore, for some $c < c_0$, there exists $\lambda < 0$ with corresponding real eigenvector which is a perturbation of the eigenvector $z_0 > 0$ satisfying $(\lambda_0^2 A + \lambda_0 c_0 I + B)z_0 = 0$ (see Kato [26], page 91). But a sufficiently small real perturbation of z_0 will also be positive, contradicting the fact that c_0 is the least velocity for which there exists a stable monotone eigenvalue. So the extensions of the eigenvalue branches for $c < c_0$ must become complex. The fact that eigenvalues of a real matrix must occur in complex conjugate pairs yields the required result.

$\square$

This gives a complete picture of the behaviour of the stable monotone eigenvalues as c is varied. To infer that the dimension of the stable manifold at T is at least $N+1$ for $c > c_0$, we show that for large c, N eigenvalues of the travelling-wave problem tend to $-\infty$. That there are at least $N+1$ eigenvalues of the travelling-wave problem in the left-half plane is then immediate by the continuous dependence of the eigenvalues on the velocity c, and the fact that no eigenvalue can cross the gap between the two stable monotone eigenvalues by Theorem 3.2.7.

Theorem 3.2.9 *Let $B \in P^{N\times N}$ with $\mu_{PF}(B) > 0$. Then as $c \to \infty$, N eigenvalues of the travelling-wave problem tend to $-\infty$.*

Proof. Suppose that $c > 0$, $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^N\backslash\{0\}$ satisfy $(\lambda^2 A + \lambda c I + B)z = 0$. Dividing by c^2 and setting $\mu = \lambda c^{-1}$ and $\epsilon = c^{-2}$ leads to $(\mu^2 A + \mu I + \epsilon B)z = 0$ and this can be written in the form

$$\begin{pmatrix} 0 & I \\ -\epsilon A^{-1}B & -A^{-1} \end{pmatrix} \begin{pmatrix} z \\ \mu z \end{pmatrix} = \mu \begin{pmatrix} z \\ \mu z \end{pmatrix}.$$

When $\epsilon = 0$, the set of eigenvalues μ is comprised of zero, with algebraic multiplicity N, and the N eigenvalues of $-A^{-1}$. Thus by the continuous dependence of the eigenvalues μ on the parameter ϵ, N eigenvalues μ must tend to finite negative limits as $\epsilon \downarrow 0$ since A is positive-definite. By definition of μ and ϵ, the corresponding λ tend to $-\infty$ as $c \to \infty$.

□

Thus for $c > c_0$, the stable manifold at T has dimension at least $N+1$, so $D_s(T) + D_u(S) \geq 2N+1$. For completeness, the behaviour of the eigenvalues of the travelling-wave problem having real part greater than the stable monotone eigenvalue closer to zero for large c can be elucidated under some additional restrictions on the matrix B. By the proof of Theorem 3.2.9, N eigenvalues of the travelling-wave problem tend to $-\infty$ as $c \to \infty$, and there are N eigenvalues λ such that $\lambda/c \to 0$ as $c \to \infty$. But the exact number of eigenvalues in the open left-half plane for c sufficiently large in terms of the number of eigenvalues of B in the right-half plane can be obtained from the Implicit Function Theorem.

Theorem 3.2.10 *Let $B \in P^{N\times N}$ with $\mu_{PF}(B) > 0$ have only simple eigenvalues, none of which is imaginary, and k of which are in the open right-half plane. Then for c sufficiently large, there are $N+k$ eigenvalues of the travelling-wave problem in the left-half plane.*

Proof. Let β be an eigenvalue of B with corresponding eigenvector x, $\| x \|_{\mathbb{C}^N} = 1$. If $c > 0, \lambda$ and z satisfy

$$(\lambda^2 A + \lambda c I + B)z = 0, \tag{3.9}$$

set $\epsilon = c^{-2}$ and $\mu = \lambda c^{-1}$ so that

$$(\mu^2 A + \mu I + \epsilon B)z = 0. \tag{3.10}$$

Now for $\epsilon \neq 0$, define $\delta \in \mathbb{C}$ via $\mu = \epsilon(-\beta + \delta)$. Then (3.10) becomes

$$(\epsilon^2(-\beta+\delta)^2 A + \epsilon(-\beta+\delta)I + \epsilon B)z = 0$$

and cancelling ϵ gives

$$(\epsilon(-\beta+\delta)^2 A + (-\beta+\delta)I + B)z = 0. \tag{3.11}$$

Note that (3.11) is satisfied when $\epsilon = \delta = 0$ and $z = x$.

Since (3.11) is equivalent to (3.9) when $\epsilon \neq 0$, the Implicit Function Theorem applied to (3.11) gives the behaviour of an eigenvalue λ at large c.

Let $F : \mathbb{R} \times \mathbb{C} \times \mathbb{C}^N \to \mathbb{C} \times \mathbb{C}^N$ be defined by

$$F(\epsilon, \delta, z) = \begin{cases} \langle z, x\rangle_{\mathbb{C}^N} - 1 \\ (\epsilon(-\beta+\delta)^2 A + (-\beta+\delta)I + B)z. \end{cases}$$

Then $F(0,0,x) = 0$, F is continuously differentiable and

$$d_{(\delta,z)}F[0,0,x](\delta, z) = \begin{cases} \langle z, x\rangle_{\mathbb{C}^N} \\ (B - \beta I)z + \delta x. \end{cases}$$

If $d_{(\delta,z)}F[0,0,x](\delta,z) = 0$, then $(B-\beta I)z + \delta x = 0$. If $\delta \neq 0$, $x \in \text{range}(B-\beta I)$. But since $x \in \ker(B - \beta I)$, this contradicts the fact that β is a simple eigenvalue of B. Also $(B - \beta I)z = 0$, $z \neq 0$, means z is a non-zero multiple of x, which contradicts $\langle z, x\rangle_{\mathbb{C}^N} = 0$. Hence $(\delta, z) = (0, 0)$ and thus $d_{(\delta,z)}F[0, 0, x]$ is injective and thus invertible. The Implicit Function Theorem then implies that $F(\epsilon, \delta, z) = 0$ defines δ and z as continuously differentiable functions of ϵ on a neighbourhood of $\delta = 0, z = x, \epsilon = 0$. In particular, for $\epsilon > 0$ small, δ is close to zero so that for the corresponding λ and c satisfying (3.9), $\lambda = 0$ when $1/c = 0$ and $\lambda = \frac{1}{c}(-\beta+\delta)$ is approximately a small multiple of $-\beta$ for $1/c > 0$ and small; that is, λ moves into the opposite half plane to β as c decreases from infinity.

It follows that, for c sufficiently large, in a small neighbourhood of the origin, there are exactly k eigenvalues of the travelling-wave problem which have negative real part. In addition to the N eigenvalues that we know from Theorem 3.2.9 to tend to $-\infty$, this gives $N + k$ eigenvalues in the left-half plane.

$\square$

Remark. Eigenvalues of the travelling-wave problem *can* cross the imaginary axis as c is varied in the monostable case. Examples of this have been observed in numerical experiments.

Chapter 4

The existence of additional zeros

In this chapter, we study the existence of additional equilibria of a locally monotone function $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, for which it is known that $f(S) = f(T) = 0$ for some $S, T \in \mathbb{R}^N$ with $S < T$. (The expression 'locally monotone' is defined in Chapter 1.) Our main tool is Brouwer degree. First, existence of an intermediate equibrium is proved for f bistable, and then for f unstable. We then address the stability of these intermediate zeros of f. The chapter concludes with a geometrical application of these results, in the case where $f = \nabla\phi$ for a scalar-valued function ϕ.

First some notation. As usual, let $S, T \in \mathbb{R}^N$ with $S < T$, and let C denote the open N-rectangle (S, T). We write $B_\delta(x)$ for the open ball in $\mathbb{R}^N$, centre $x \in \mathbb{R}^N$ and radius δ. For $f : \mathbb{R}^N \to \mathbb{R}^N$ continuous, Ω an open bounded subset of $\mathbb{R}^N$ and $p \in \mathbb{R}^N \backslash f(\partial\Omega)$, let $\deg(f, \Omega, p)$ represent the Brouwer degree of f at p relative to Ω, and $\text{ind}(f, x_0, p) := \deg(f, B_\delta(x_0), p)$ for $\delta > 0$ sufficiently small, the index of an isolated solution, x_0, of $f(x) = p$. If f is continuously differentiable and $df[x_0]$ is invertible at an isolated solution x_0, then $\text{ind}(f, x_0, p) = \text{sgn}(\det(df[x_0]))$, where $\text{sgn}(\alpha) = 1$ or -1 according to whether α is positive or negative. (See Deimling [14], for example.) Recall that $\det A = \Pi_{i=1}^N \lambda_i$ for $A \in M^{N\times N}$ with (complex) eigenvalues $\lambda_1, \ldots, \lambda_N$, listing an eigenvalue k times when it has algebraic multiplicity k.

In [42], Vol'pert and Vol'pert prove the following vital result on the sign of a locally monotone function.

Lemma 4.0.11 *Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be locally monotone. If, for some k,*

$1 \leq k \leq N$, $f_k(x) = 0$ *and* $y \geq x$, $y \neq x$ *and* $y_k = x_k$, *then* $f_k(y) > 0$. *Also,* $f_k(y) < 0$ *if* $y \leq x$, $y \neq x$ *and* $y_k = x_k$.

Proof. Let $\phi(t) = f_k(x(1-t)+yt)$, where $x \neq y$ and $x_k = y_k$. If $\phi(t) = 0$, local monotonicity gives that $\phi'(t) > 0$ (< 0) when $y \geq x$ ($y \leq x$). Since $\phi(0) = 0$, the result follows in both cases.

$\square$

4.1 Existence of additional zeros for bistable and unstable functions

Theorem 4.1.1 *Let* $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *be a locally monotone function such that* S *and* T *are stable equilibria of* f. *Then there exists* $x \in C$ *with* $f(x) = 0$.

Proof. Let $w \in C$ be fixed and let $h : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ be defined by $h(t,x) = (1-t)(w-x) + tf(x)$. Since $f(S) = f(T) = 0$, Lemma 4.0.11 gives that for every k, $1 \leq k \leq N$,

$$f_k(x) > 0 \text{ for } x \in \partial C \backslash \{S\} \text{ with } x_k = S_k \tag{4.1}$$

and

$$f_k(x) < 0 \text{ for } x \in \partial C \backslash \{T\} \text{ with } x_k = T_k. \tag{4.2}$$

Now for each k and $x \in \{y \in \partial C : y_k = S_k, y \neq S\}$,

$$w_k - x_k > 0 \text{ and } f_k(x) > 0, \tag{4.3}$$

so $f(x) \neq \lambda(x-w)$ for any $\lambda > 0$. Similarly, for $x \in \{y \in \partial C : y_k = T_k, y \neq T\}$,

$$w_k - x_k < 0 \text{ and } f_k(x) < 0, \tag{4.4}$$

so $f(x) \neq \lambda(x-w)$ for any $\lambda > 0$. Hence given $x \in \partial C \backslash \{S,T\}$ and $t \in (0,1)$, $f(x) \neq \frac{1-t}{t}(x-w)$; i.e. $h(t,x) \neq 0$. Also, $h(0,x) = w - x \neq 0$ as $w \notin \partial C$ and $h(1,x) = f(x) \neq 0$ since, for some k, $f_k(x) \neq 0$. Therefore

$$h(t,x) \neq 0 \text{ for } (t,x) \in [0,1] \times \{\partial C \backslash \{S,T\}\}. \tag{4.5}$$

Note that $f(S) = f(T) = 0$ and $S, T \in \partial C$, so $\deg(f, C, 0)$ is not defined. We now show there are neighbourhoods N_S and N_T of S and T such that $h(t, x) \neq 0$ for $(t, x) \in [0, 1] \times \{(N_S \backslash C) \cup (N_T \backslash C)\}$.

Suppose that there is a sequence $\{x^i\}$ with $\{x^i - S\} \subset \mathbb{R}^N \backslash \mathbb{R}^N_+$, $\| x^i - S \| < 1/i$ and

$$(1 - t_i)(w - x^i) + t_i f(x^i) = 0 \text{ for some } t_i \in [0, 1]. \tag{4.6}$$

Since f is differentiable at S and $f(S) = 0$,

$$f(x^i) = L(x^i - S) + R(x^i) \tag{4.7}$$

where $L := df[S]$ and $R : \mathbb{R}^N \to \mathbb{R}^N$ is such that $\| R(x^i) \| / \| x^i - S \| \to 0$ as $x^i \to S$. Hence

$$0 = t_i \left[L \frac{(x^i - S)}{\| x^i - S \|} + \frac{R(x^i)}{\| x^i - S \|} \right] + \frac{(1 - t_i)}{\| x^i - S \|}(w - x^i) \tag{4.8}$$

since $x^i - S \neq 0$ for each $i \in \mathbb{N}$. Now $\{s^i\} := \left\{ \frac{x^i - S}{\| x^i - S \|} \right\}$, a bounded sequence in $\mathbb{R}^N$, has a convergent subsequence, say $s^j \to s$; L is continuous, and so $Ls^j \to Ls$ as $j \to \infty$. Also, $\{t_j\}$ is bounded in $\mathbb{R}$, so has a subsequence $t_m \to t$, and since $\frac{R(x^m)}{\| x^m - S \|} \to 0$ as $m \to \infty$, we see that

$$\frac{(1 - t_m)}{\| x^m - S \|}(w - x^m) = -t_m \left[L \frac{(x^m - S)}{\| x^m - S \|} + \frac{R(x^m)}{\| x^m - S \|} \right]$$

tends to a limit as $m \to \infty$. Since $\| x^m - S \| \to 0$, it follows that $t_m \to 1$. Define

$$\beta = \lim_{m \to \infty} \frac{(1 - t_m)}{\| x^m - S \|} \geq 0. \tag{4.9}$$

Then $0 = Ls + \beta(w - S)$ where $\beta \geq 0$ and $s \notin \left(\mathbb{R}^N_+\right)^o$ has $\| s \| = 1$. But zero is not an eigenvalue of L. So $\beta > 0$, and for some $\beta(w - S) \in \left(\mathbb{R}^N_+\right)^o$ and $-s \notin -\left(\mathbb{R}^N_+\right)^o$,

$$L(-s) = \beta(w - S). \tag{4.10}$$

Moreover, L is continuous and $\left(\mathbb{R}^N_+\right)^o$ is open, so there exist $\tilde{u} \notin -\mathbb{R}^N_+$ and

$\tilde{v} \in \left(\mathbb{R}_+^N\right)^o$ such that $L\tilde{u} = \tilde{v}$. Now for some $\gamma > 0$, $\tilde{v} \geq \gamma\tilde{u}$, and hence $L\tilde{u} \geq \gamma\tilde{u}$. Since f is locally monotone, $L \in P^{N\times N}$. So the matrix $L+\alpha I$ has positive entries for some $\alpha > 0$ and

$$(L + \alpha I)\tilde{u} \geq (\gamma + \alpha)\tilde{u} \tag{4.11}$$

where $\tilde{u} \notin -\mathbb{R}_+^N$. Theorem 2.2.3 then implies that $L + \alpha I$ has a real eigenvalue greater than $\gamma+\alpha$, whence L has a real eigenvalue greater than γ. This contradicts the hypothesis that all the eigenvalues of $L = df[S]$ are in the open left-half plane. Thus there exists $k \in \mathbb{N}$, such that

$$x - S \notin \mathbb{R}_+^N,\ \| x - S \| < 1/k \ \Rightarrow\ h(t,x) \neq 0 \text{ for each } t \in [0,1] \tag{4.12}$$

A similar argument shows that k may be chosen so that, in addition,

$$x - T \notin -\mathbb{R}_+^N,\ \| x - T \| < 1/k \Rightarrow\ h(t,x) \neq 0 \text{ for each } t \in [0,1]. \tag{4.13}$$

Hence there are open neighbourhoods N_S and N_T of S and T respectively such that

$$x \in (\partial N_S) \cup (\partial N_T),\ h(t,x) = 0 \ \Rightarrow\ x \in C, \tag{4.14}$$

the case of $x \in \partial C$ having been previously excluded. Next note that S and T are isolated zeros of f by the Inverse Function Theorem. We can therefore take N_S and N_T sufficiently small to ensure that there are no additional zeros of f in $\overline{N_S} \cup \overline{N_T}$. Furthermore,

$$\text{ind}(f, S, 0) = \text{ind}(f, T, 0) = (-1)^N \tag{4.15}$$

since for $x = S$ or T, $\text{ind}(f,x,0) = \text{sgn}(\det(df[x]))$, and all the eigenvalues of $df[S]$ and $df[T]$ are in the open left-half plane. (Note that non-real eigenvalues of a real matrix occur in complex conjugate pairs.)

Now let $\Omega = C \cup N_S \cup N_T$. Then $h(t,x) \neq 0$ for any $t \in [0,1], x \in \partial\Omega$, and so since $w \in C \subset \Omega$, the homotopy property of Brouwer degree gives that

$$\deg(f, \Omega, 0) = \deg(w - I, \Omega, 0) = (-1)^N, \tag{4.16}$$

where I denotes the identity mapping. If $\Omega_1 = \Omega\backslash\{\overline{C} \cap (\partial N_S \cup \partial N_T)\}$, then

$\deg(f, \Omega, 0) = \deg(f, \Omega_1, 0)$ by the excision property, so

$$\begin{aligned} (-1)^N = \deg(f, \Omega_1, 0) &= \operatorname{ind}(f, S, 0) + \operatorname{ind}(f, T, 0) + \deg(f, \Omega_1 \backslash (N_S \cup N_T), 0) \\ &= 2(-1)^N + \deg(f, \Omega_2, 0), \end{aligned}$$

where $\Omega_2 = \Omega_1 \backslash (N_S \cup N_T) \subset C$. Hence $\deg(f, \Omega_2, 0) = (-1)^{N+1} \neq 0$, and thus by the existence property of Brouwer degree, there exists $x \in \Omega_2 \subset C$ with $f(x) = 0$, as required.

□

The corresponding result in the case in which S and T are unstable equilibria is proved similarly, using Theorem 2.2.5 instead of Theorem 2.2.3.

Theorem 4.1.2 *Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be a locally monotone function such that S and T are unstable equilibria of f. Then there exists $x \in C$ with $f(x) = 0$.*

Proof. As in Theorem 4.1.1, the function $h : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ defined as above is non-zero on $\partial C \backslash \{S, T\}$. In this case, the existence of neighbourhoods N_S and N_T of S and T, with $h(t, x) \neq 0$ for $x \in (N_S \cup N_T) \cap C$ and $t \in [0,1]$ is established as follows. If there exists a sequence $\{t_i, x^i\} \subset [0,1] \times C$ with $0 <\| x^i - S \|< 1/i$ such that $t_i f(x^i) + (1 - t_i)(w - x^i) = 0$, it follows as previously that $Ls = -\beta(w - S)$, where $L := df[S]$, $s \in \mathbb{R}^N_+$, $w - S \in \left(\mathbb{R}^N_+\right)^o$ and $\beta > 0$. Theorem 2.2.5 applies to give $\mu_{PF}(L) < 0$ since $Ls < 0$, which contradicts that L has an eigenvalue in the closed right-half plane. Similar arguments apply at T, so that a homotopy from f to $w - I$ on the boundary of $\Omega := C \backslash (N_S \cup N_T)$ for some sufficiently small neighbourhoods N_S and N_T, yields that $\deg(f, \Omega, 0) = \deg(w - I, \Omega, 0) = (-1)^N$. Hence there exists some $x \in \Omega \subset C$ with $f(x) = 0$.

□

4.2 Stability of intermediate zeros

To comment on the stability of intermediate equilibria, additional hypotheses on f are needed.

Theorem 4.2.1 *Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be a locally monotone function whose zeros are isolated. Suppose also that when x is a zero of f, $\mu_{PF}(df[x]) \neq 0$. Then if both S and T are stable (unstable) equilibria, then there exists an unstable (stable) equilibrium in (S, T). Furthermore, if $\{x^1, \ldots, x^k\}$ is a maximal totally-ordered set of equilibria in the interval (S, T), then k is odd and x^i is a stable or unstable (unstable or stable) equilibrium according to whether i is even or odd.*

Proof. We shall prove the theorem in the case where S and T are stable. By Theorem 4.1.1, there exists some $x \in C$ with $f(x) = 0$, and by the hypothesis that the zeros of f are isolated, there are a finite number of such zeros in the compact set $\overline{C}$. Let $x^1, \ldots, x^k$ be a maximal totally-ordered subset of these equilibria, so that $S := x^0 < x^1 < \ldots < x^{k+1} =: T$, where there are no zeros x of f with $x^i < x < x^{i+1}$ for any $i = 0, 1, \ldots, k$. Suppose, for contradiction, that x^i is a stable equilibrium for all $i = 1, \ldots, k$. Then in particular, x^1 is stable. But Theorem 4.1.1 then implies that there exists a zero of f in the order interval (x^0, x^1), contradicting the maximality of the totally-ordered subset. Hence x^1 is an unstable equilibrium, since the Perron-Frobenius eigenvalue of $df[x^1]$ is non-zero. It follows that any two equilibria $x^i < x^{i+1}$, $i = 0, 1, \ldots, k$, must be equilibria of opposite type, from which the rest of the theorem is immediate.

□

Remark. The proof of Theorem 4.2.1 uses the fact that the local monotonicity of f implies that $df[x] \in P^{N \times N}$ for each x with $f(x) = 0$, since Theorems 4.1.1 and 4.1.2 thus apply to the interval (x, y) whenever x and y are zeros of f with $x < y$. However, when S and T are both stable equilibria, the proof of Theorem 4.1.1 needs only conditions on f on ∂C to yield that $\deg(f, \Omega_2, 0) = (-1)^{N+1}$, where $\Omega_2 \subset C$. If the zeros of f are isolated, this implies that there is an equilibrium of f between S and T that is not stable. (Note that it cannot be inferred from $\operatorname{ind}(f, x, 0) = (-1)^N$ that x is not an unstable equilibrium of f, and hence we cannot deduce the analogous result when S and T are both unstable equilibria.) This weakening of the hypotheses is useful in the following corollary; the case when $f = \nabla\phi$.

Corollary 4.2.2 *Suppose that $\phi : \mathbb{R}^N \to \mathbb{R}$ is a C^2-function whose critical points are non-degenerate (that is, the Hessian of ϕ is invertible at each such point), and that*

(i) *ϕ has local maxima at S and T*

(ii) *At points x of ∂C at which a normal to ∂C exists, $\frac{\partial \phi}{\partial n} := \langle \nabla \phi, n(x) \rangle < 0$, where $n(x)$ is the outward normal to ∂C at x.*

Then there exists a critical point x of ϕ in $\overline{C} \backslash \{S, T\}$. If $x \in C$, x is not a local maximum; if additionally the space dimension n is even, then x is a saddle point.

Proof. By the continuity of $\nabla \phi$, condition (ii) implies that *non-strict* inequalities for $\frac{\partial \phi}{\partial x^k}$ hold for all $x \in \partial C$ with $x^k = S_k$ or T_k. If there are critical points other than S and T on ∂C, then the result is clearly true. Otherwise, taking $f = \nabla \phi$ in the first part of the proof of Theorem 4.1.1 gives that $h(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \{\partial C \backslash \{S, T\}\}$. Now (i) and the fact that the critical points of ϕ are non-degenerate imply that all the eigenvalues of the Hessian matrices at S and at T are in the open left-half plane. Also, (ii) yields that the off-diagonal elements of the Hessian matrices at S and T are non-negative. Following the method of the proof of Theorem 4.1.1, this suffices to give that there exists a critical point x of ϕ in C with $\text{ind}(\nabla \phi, x, 0) = (-1)^{N+1}$, since the critical points of ϕ are isolated by the Inverse Function Theorem. Denoting the Hessian of ϕ at x by $d^2\phi[x]$, we get that $\text{sgn}(\det(d^2\phi[x])) = (-1)^{N+1}$ since $d^2\phi[x]$ is invertible. The fact that $\det A = \Pi_{i=1}^{N} \lambda_i$ for an $N \times N$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_N$ gives the remaining observations.

□

Chapter 5

Degree theory for the analysis of travelling waves

The remainder of this thesis is devoted to the proof of the existence of travelling-wave solutions for gradient-dependent parabolic systems, as introduced briefly in Chapter 1. In the present chapter, we begin with some technical machinery on weighted function spaces and degree theory. We then turn to a precise formulation of the approximate travelling-wave problem (1.10) in this language. The rest of the chapter is concerned with the non-trivial task of showing that operators associated with (1.10) lie in the domain of definition of the degree.

Henceforth, we suppose the following. Let $A \in M^{N\times N}$ be a positive-definite diagonal matrix, and let $S, T \in \mathbb{R}^N$ be such that $S < T$.
Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be such that

(f1) f is locally monotone (as defined in Chapter 1);

(f2) $f(S) = f(T) = 0$;

(f3) $\mu_{PF}(df[S]), \mu_{PF}(df[T]) < 0$, (that is, f is *bistable*);

(f4) there is a non-zero finite number m of zeros u^k of f with $S < u^k < T$ $(k = 1, \dots, m)$; furthermore, for each such u^k,

$$\mu_{PF}(df[u^k]) > 0.$$

Lastly, let $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ be such that

(G1) G is diagonal-matrix valued;

(G2) there exist $\beta, \gamma \in C(\mathbb{R}, \mathbb{R}^+)$ such that for each $u, v \in \mathbb{R}^N$,

$$||G(u,v)v|| \leq \beta(u) + \gamma(u)||v||;$$

(G3) $G(u,0) = 0$ for each $u \in \mathbb{R}^N$ with $f(u) = 0$.

The following consequence of **(G2)** will prove useful.

Lemma 5.0.3 *Let Ω be a bounded subset of $\mathbb{R}^N$ and let $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ satisfy* **(G2)**. *Then there exists a positive constant α^*, depending on Ω, such that for each $u \in \Omega$,*

$$||G(u,v)v|| \leq \alpha^* ||v||$$

for every $v \in \mathbb{R}^N$.

Proof. As G is continuous, there exists $\alpha_0 > 0$ such that

$$||G(u,v)|| \leq \alpha_0 \tag{5.1}$$

for each $u \in \Omega$ and $v \in \mathbb{R}^N$ with $||v|| \leq 1$. Also, if $v \in \mathbb{R}^N$ has $||v|| \geq 1$, then by **(G2)**,

$$||G(u,v)v|| \leq \beta(u) + \gamma(u)||v|| \leq [\beta(u) + \gamma(u)]||v|| \tag{5.2}$$

for each $u \in \mathbb{R}^N$. Thus (5.1) and (5.2) yield that

$$||G(u,v)v|| \leq \max\{\alpha_0, \beta(u) + \gamma(u)\}||v|| \tag{5.3}$$

for each $u \in \Omega, v \in \mathbb{R}^N$. The result follows.

□

5.1 Technical machinery

5.1.1 Weighted function spaces

Let $\mu(s) := 1 + s^2$ $(s \in \mathbb{R})$. Let Ω be an open subset of $\mathbb{R}$. Then we define the *weighted* space $L_{2,\mu}(\Omega, \mathbb{R}^N)$ to be the linear space of equivalence classes of measurable functions $u : \mathbb{R} \to \mathbb{R}^N$ such that

$$\|u\|_{L_{2,\mu}(\Omega,\mathbb{R}^N)} := \langle u, u\rangle^{\frac{1}{2}}_{L_{2,\mu}(\Omega,\mathbb{R}^N)} < \infty, \tag{5.4}$$

with inner product

$$\langle u, v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)} := \int_\Omega \langle u(s), v(s)\rangle \mu(s) ds, \ (u, v \in L_{2,\mu}(\Omega, \mathbb{R}^N)). \tag{5.5}$$

The weighted Sobolev space $W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ is then defined to be the subspace of $L_{2,\mu}(\Omega, \mathbb{R}^N)$ comprising of functions $u \in L_{2,\mu}(\Omega, \mathbb{R}^N)$ for which $\mathcal{D}u \in L_{2,\mu}(\Omega, \mathbb{R}^N)$. (Recall from Section 2.1 that $\mathcal{D}u$ denotes the vector of weak derivatives of the components of u.) This space is endowed with the inner product

$$\langle u, v\rangle_{W^1_{2,\mu}(\Omega,\mathbb{R}^N)} := \langle u, v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)} + \langle \mathcal{D}u, \mathcal{D}v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)}, \ (u, v \in W^1_{2,\mu}(\Omega, \mathbb{R}^N)) \tag{5.6}$$

and norm $\|\cdot\|_{W^1_{2,\mu}(\Omega,\mathbb{R}^N)}$.
For notational convenience, we make the following definitions;

$$\omega(s) := \sqrt{\mu(s)} \text{ and } \nu(s) := \mu^{-1}(s)\mu'(s), \quad s \in \mathbb{R}. \tag{5.7}$$

Lemma 5.1.1 *The linear multiplication operators $M : W^1_2(\Omega, \mathbb{R}^N) \to W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ and $N : W^1_{2,\mu}(\Omega, \mathbb{R}^N) \to W^1_2(\Omega, \mathbb{R}^N)$, defined by*

$$Mu(s) = \omega^{-1}(s)u(s) \quad (u \in W^1_2(\Omega, \mathbb{R}^N)) \text{ and} \tag{5.8}$$

$$Nu(s) = \omega(s)u(s) \qquad (u \in W^1_{2,\mu}(\Omega, \mathbb{R}^N)) \tag{5.9}$$

respectively, are bounded.

Proof. Let $u \in W^1_{2,\mu}(\Omega, \mathbb{R}^N)$. Then

$$\begin{aligned}
\|Mu\|^2_{W^1_{2,\mu}(\Omega,\mathbb{R}^N)} &= \int_\Omega \left\{ \|\omega^{-1}(s)u(s)\|^2 + \|\omega^{-1}(s)\mathcal{D}u(s) - \frac{1}{2}\omega^{-1}(s)\nu(s)u(s)\|^2 \right\} \mu(s)\, ds \\
&= \int_\Omega \left\{ \|u(s)\|^2 + \|\mathcal{D}u(s) - \frac{1}{2}\nu(s)u(s)\|^2 \right\} ds \\
&\leq \int_\Omega \left\{ \|u(s)\|^2 + 2\|\mathcal{D}u(s)\|^2 + \frac{1}{2}\|\nu(s)u(s)\|^2 \right\} ds \\
&\leq \int_\Omega \left\{ \frac{3}{2}\|u(s)\|^2 + 2\|\mathcal{D}u(s)\|^2 \right\} ds \\
&\leq 2\|u\|^2_{W^1_2(\Omega,\mathbb{R}^N)},
\end{aligned}$$

using the fact that $|\nu(s)| = \dfrac{2|s|}{1+s^2} \leq 1$ for each $s \in \mathbb{R}$. The result follows for M; the argument for N is similar.

$\square$

Using Lemma 5.1.1, we deduce from the corresponding properties of $W^1_2(\Omega, \mathbb{R}^N)$ (see section 2.1) that $W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ is a separable Hilbert space. It also follows that $C^\infty_0(\mathbb{R}, \mathbb{R}^N)$ is dense in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. This enables proof of the following vital estimate.

Lemma 5.1.2 *For $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,*

$$\|u(s)\| \leq \frac{\|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}}{\sqrt{\mu(s)}} \tag{5.10}$$

for each $s \in \mathbb{R}$.

Proof. Let $u \in C^\infty_0(\mathbb{R}, \mathbb{R}^N)$. Then for $s \leq 0$,

$$\begin{aligned}
\|u(s)\|^2 &= 2\int_{-\infty}^s \langle u(t), u'(t)\rangle\, dt \\
&= 2\int_{-\infty}^s \langle u(t), u'(t)\rangle \mu(t) \frac{1}{\mu(t)}\, dt \\
&\leq \frac{2}{\mu(s)} \int_{-\infty}^s \|u(t)\|\|u'(t)\|\mu(t)\, dt \\
&\leq \frac{1}{\mu(s)} \int_{-\infty}^s \left\{ \|u(t)\|^2 + \|u'(t)\|^2 \right\} \mu(t)\, dt \\
&\leq \frac{\|u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}}{\mu(s)}.
\end{aligned}$$

A similar argument, integrating from s to ∞, shows that (5.10) holds for $s > 0$. Now $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and if a sequence converges in $L_2(\mathbb{R}, \mathbb{R}^N)$, then a subsequence converges pointwise almost everywhere. Hence (5.10) holds for almost every $s \in \mathbb{R}$. Since

$$W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \subset W^1_2(\mathbb{R}, \mathbb{R}^N) \subset C(\mathbb{R}, \mathbb{R}^N),$$

the result follows.

$\square$

5.1.2 Degree theory for $(S)_+$ operators

A degree theory applicable here is that for operators of class $(S)_+$, as developed in [40, 7, 3], and discussed in [42].

Let X denote a real, reflexive, separable Banach space, and let $W : X \to X^*$. Then W is of class $(S)_+$, or $W \in (S)_+$, if for $\{u_n\} \subset X$,

$$\left.\begin{array}{r} u_n \rightharpoonup u_o \text{ in } X \\ \text{and } \limsup\limits_{n\to\infty} (W(u_n))(u_n - u_o) \leq 0 \end{array}\right\} \Rightarrow u_n \to u_o \text{ in } X. \tag{5.11}$$

Suppose that W is also bounded and demicontinuous (continuous from the strong topology of X to the weak topology of X^*). Let Ω be a bounded open subset of X; denote the closure and boundary of Ω by $\overline{\Omega}$ and $\partial\Omega$ respectively. If

$$0 \notin W(\partial\Omega),$$

then an integer-valued degree, $deg_{(S)_+}(W, \Omega, 0)$, is defined via Galerkin approximations.

The usual properties of a degree function hold. If $deg_{(S)_+}(W, \Omega, 0) \neq 0$, then there exists $u \in \Omega$ with $W(u) = 0$. Also, the degrees of functions which are $(S)_+$-homotopic relative to Ω are equal; if $W_1, W_2 : X \to X^*$ are bounded demicontinuous $(S)_+$ mappings, W_1 and W_2 are said to be $(S)_+$- *homotopic relative to* Ω if there is a bounded demicontinuous operator $W : \overline{\Omega} \times [0,1] \to X^*$ such that

$$(i) \qquad 0 \notin W(\partial\Omega \times [0,1]) \tag{5.12}$$

(ii) $\quad W(u,0) = W_1(u), W(u,1) = W_2(u)$ for all $u \in \overline{\Omega}$, and $\quad$ (5.13)

(iii) $\quad$ For $\{u_n, t_n\} \subset \overline{\Omega} \times [0,1]$,

$$\left.\begin{array}{c} u_n \rightharpoonup u_o \text{ in } X, \\ t_n \to t_o, \text{ and} \\ \limsup\limits_{n\to\infty} (W(u_n,t_n))(u_n - u_o) \le 0 \end{array}\right\} \Rightarrow u_n \to u_o \text{ in } X. \tag{5.14}$$

5.2 Preliminaries for the approximate system

5.2.1 Formulation of the problem

Let $R > 0$ and $\sigma_R \in C^1(\mathbb{R}, [0,1])$ be such that

(i) $\quad \sigma_R(s) = 1$ when $|s| \le R$,

(ii) $\quad \operatorname{supp} \sigma_R \subset [-R-1, R+1]$,

(iii) $\quad \{\sigma_R\}$ and $\{\sigma_R'\}$ are each uniformly bounded and equicontinuous as families parametrised by R.

We seek a constant c and a function $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that

$$Aw''(s) + cw'(s) + \sigma_R(s)G(w(s), w'(s))w'(s) + f(w(s)) = 0, \; s \in \mathbb{R}, \tag{5.15}$$

$$w'(s) > 0, \; s \in \mathbb{R}, \tag{5.16}$$

and

$$w(s) \to S \text{ as } s \to -\infty, \;\; w(s) \to T \text{ as } s \to \infty. \tag{5.17}$$

Let

$$\psi(s) = T\alpha(s) + S(1 - \alpha(s)), \tag{5.18}$$

and seek w in the form

$$w(s) = u(s) + \psi(s), \tag{5.19}$$

where $\alpha \in C^\infty(\mathbb{R}, [0,1])$ is an arbitrary but fixed monotone function with $\alpha(s) = 0$ when $s \le -1$ and $\alpha(s) = 1$ when $s \ge 1$. Then w satisfies (5.15), (5.16) and (5.17) if and only if u satisfies

$$A(u''(s)+\psi''(s))+c[u'(s)+\psi'(s)]+\sigma_R(s)G(u(s)+\psi(s),u'(s)+\psi'(s))(u'(s)+\psi'(s))$$

$$+ f(u(s)+\psi(s)) = 0, \tag{5.20}$$

$$u'(s)+\psi'(s) > 0, \quad s \in \mathbb{R} \tag{5.21}$$

and

$$u(s) \to 0 \text{ as } s \to \pm\infty. \tag{5.22}$$

Following [42], we seek a solution u to (5.20), (5.21) and (5.22) in the weighted Sobolev space $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ defined in section 5.1.1. Note that estimate (5.10) implies that (5.22) is satisfied for *every* $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$. The fact that the constant c is not known *a priori* is overcome by functionalising the parameter, as follows.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(s) := \begin{cases} e^s & \text{, if } s < 0 \\ 1 & \text{, if } s \geq 0. \end{cases} \tag{5.23}$$

For $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, set

$$\rho(u) := \left(\int_{\mathbb{R}} \|u(s)+\psi(s)-T\|^2 \phi(s)\, ds\right)^{\frac{1}{2}} \tag{5.24}$$

and

$$c(u) := \log\, \rho(u), \tag{5.25}$$

where ψ is defined in (5.18). Define

$$u_h(s) := u(s+h)+\psi(s+h)-\psi(s), \; s \in \mathbb{R}, \tag{5.26}$$

for each $h \in \mathbb{R}$. Then $c(\cdot)$ has the following properties.

Lemma 5.2.1 *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Then $c(u_h)$ is monotone in h, and*

$$c(u_h) \to \pm\infty \text{ as } h \to \mp\infty.$$

Proof. By definition,

$$\begin{aligned} \rho(u_h) &= \Big(\int_{\mathbb{R}} \|u(s+h) + \psi(s+h) - T\|^2 \phi(s)\, ds\Big)^{\frac{1}{2}} \\ &= \Big(\int_{\mathbb{R}} \|u(s) + \psi(s) - T\|^2 \phi(s-h)\, ds\Big)^{\frac{1}{2}}. \end{aligned}$$

That $c(u_h)$ is a decreasing function of h follows from the monotonicity of ϕ and of the logarithmic function. The monotonicity of ϕ implies that for each $s \in \mathbb{R}$ and $h \geq 0$,

$$\|u(s) + \psi(s) - T\|^2 \phi(s-h) \leq \|u(s) + \psi(s) - T\|^2 \phi(s),$$

and hence by Lebesgue's Dominated Convergence Theorem, $\rho(u_h) \to 0$ as $h \to \infty$. Levi's Monotone Convergence Theorem yields that $\rho(u_h) \to \infty$ as $h \to -\infty$. Hence $c(u_h) \to \mp\infty$ as $h \to \pm\infty$, as required.

□

Lemma 5.2.2 *The functional c is Lipschitz continuous on bounded subsets of $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.*

Proof. Let $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ be bounded. We show first that there exists $\delta > 0$ such that $\rho(u) \geq \delta$ for every $u \in \Omega$.
For any $M > 1$,

$$\rho(u) \geq \Big(\int_{-M}^{-1} \|u(s) + \psi(s) - T\|^2 \phi(s)\, ds\Big)^{\frac{1}{2}}.$$

So by the monotonicity of ϕ and definition (5.18), it follows that

$$\rho(u) \;\geq\; (\phi(-M))^{\frac{1}{2}} \Big(\int_{-M}^{-1} \|u(s) + S - T\|^2\, ds\Big)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\geq ((\phi(-M))^{\frac{1}{2}} \left[\left(\int_{-M}^{-1} \|S-T\|^2 \, ds \right)^{\frac{1}{2}} - \left(\int_{-M}^{-1} \|u(s)\|^2 \, ds \right)^{\frac{1}{2}} \right] \\
&\geq (\phi(-M))^{\frac{1}{2}} \left[(M-1)^{\frac{1}{2}} \|S-T\| - \|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \right].
\end{aligned}
$$

Choose $M_o > 1$ sufficiently large that for $u \in \Omega$,

$$
(M_o - 1)^{\frac{1}{2}} \|S-T\| - \|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} > 1.
$$

Then for $u \in \Omega$,

$$
\rho(u) \geq (\phi(-M_o))^{\frac{1}{2}} > 0. \tag{5.27}
$$

Now for every $x, y \in \mathbb{R}$, $x, y > 0$, the Mean Value Theorem yields that

$$
|\log x - \log y| \leq \frac{1}{w} |x-y|
$$

for some w between x and y. Thus it follows from (5.27) that given $u_1, u_2 \in \Omega$,

$$
\begin{aligned}
|c(u_1) - c(u_2)| &= |\log \rho(u_1) - \log \rho(u_2)| \\
&\leq (\phi(-M_o))^{-\frac{1}{2}} |\rho(u_1) - \rho(u_2)|. \qquad (5.28)
\end{aligned}
$$

For any $u_1, u_2 \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,

$$
|\rho(u_1) - \rho(u_2)| \leq \left(\int_{\mathbb{R}} \|u_1(s) - u_2(s)\|^2 \phi(s) \, ds \right)^{\frac{1}{2}}
$$

by, for example, the triangle inequality in $L_{2,\phi}(\mathbb{R}, \mathbb{R}^N)$ (where this space is defined analogously to $L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$). Hence since $0 \leq \phi(s) \leq \mu(s)$ for each $s \in \mathbb{R}$,

$$
|\rho(u_1) - \rho(u_2)| \leq \|u_1 - u_2\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}. \tag{5.29}
$$

The result follows from (5.28) and (5.29).

□

Corollary 5.2.3 *The functional c is bounded on bounded subsets of* $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.

Proof. This is immediate from Lemma 5.2.2. □

Now consider the equation

$$A(u''(s)+\psi''(s))+c(u)[u'(s)+\psi'(s)]+\sigma_R(s)G(u(s)+\psi(s),u'(s)+\psi'(s))(u'(s)+\psi'(s))$$
$$+f(u(s)+\psi(s))=0. \tag{5.30}$$

It is clear that if $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ satisfies (5.30), then u satisfies (5.20) with $c = c(u)$. Conversely, if a function u satisfies (5.20) with velocity c, then there is some $h \in \mathbb{R}$ with $c(u_h) = c$. Thus u_h satisfies (5.30) with $\sigma_R(\cdot)$ replaced by $\sigma_R(\cdot+h)$. Whence there is an equivalence between (5.20) and (5.30). Henceforth, a solution of (5.30) and (5.21) will be sought in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ (recall that (5.22) is satisfied for all functions in this space).

Our approach is to consider the following continuous deformation of (5.30):

$$A(u''+\psi'')+c(u)[u'+\psi']+\tau\sigma_R G(u+\psi,u'+\psi')(u'+\psi')+f(u+\psi)=0,\ \tau\in[0,1]. \tag{5.31}$$

When $\tau = 1$, (5.31) becomes (5.30), whilst when $\tau = 0$, (5.31) corresponds to the system (1.3) treated in [42]. We will use the known existence of a solution with non-zero degree when $\tau = 0$ to deduce existence when $\tau = 1$ using the degree theory for $(S)_+$ operators introduced in section 5.1.2. (τ is said to be the homotopy parameter.)

5.2.2 Construction of $(S)_+$ operators

To invoke the degree theory for $(S)_+$ operators, an $(S)_+$ operator associated with (5.30) is required, together with a suitable $(S)_+$ homotopy associated with (5.31). First, define $P_R : [0,1] \times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$ by

$$(P_R(\tau,u))(v) = \int_{\mathbb{R}} \langle A\mathcal{D}u(s), \mathcal{D}(v\mu)(s)\rangle\, ds\ -$$
$$\int_{\mathbb{R}} \langle \{A\psi''(s)+c(u)[\mathcal{D}u(s)+\psi'(s)]+\tau\sigma_R G(u(s)+\psi(s),\mathcal{D}u(s)+\psi'(s))[\mathcal{D}u+\psi'(s)]$$
$$+f(u(s)+\psi(s))\}, v(s)\rangle \mu(s)\, ds \tag{5.32}$$

for $\tau \in [0,1]$, $u, v \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$. (Recall from section 2.1 that $\mathcal{D}u$ is the weak derivative of u.) In order to minimise clutter, independent variables will be omitted when no confusion results.

Lemma 5.2.4 *P_R is well-defined and maps each bounded subset of $[0,1]\times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ into a bounded subset of $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$.*

Proof. Let $\tau \in [0,1], \Omega \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ be bounded, and $u \in \Omega$. The linearity of $P_R(\tau,u)$ is elementary. Let $v \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$. Use of the Cauchy-Schwarz inequality in $L_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ and the fact that $\mu^{-1}\mu'$ is uniformly bounded yields that

$$\begin{aligned}
\left|\int_{\mathbb{R}}\langle A\mathcal{D}u, \mathcal{D}(v\mu)\rangle\, ds\right| &= \left|\int_{\mathbb{R}}\langle A\mathcal{D}u, \mu^{-1}\mu' v + \mathcal{D}v\rangle \mu\, ds\right| \\
&\le \alpha_1\|\mathcal{D}u\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \alpha_2\|\mathcal{D}u\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|\mathcal{D}v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \\
&\le \alpha_3\|\mathcal{D}u\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \\
&\le \alpha_3\|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \qquad (5.33)
\end{aligned}$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ independent of u and v.

Corollary 5.2.3, together with the fact that ψ' and ψ'' have compact support, give the existence of $\alpha_4(\Omega) \in \mathbb{R}$ (dependent only on Ω), such that

$$\left|\int_{\mathbb{R}}\langle A\psi'' + c(u)[\mathcal{D}u + \psi'], v\rangle \mu\, ds\right| \le \alpha_4(\Omega)(1 + \|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)})\|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}. \qquad (5.34)$$

Now using condition **(G2)**,

$$\begin{aligned}
&\int_{\mathbb{R}}\|\tau\sigma_R G(u+\psi, \mathcal{D}u+\psi')[\mathcal{D}u+\psi']\|^2\mu\, ds \\
&\le \int_{-R-1}^{R+1}[\beta(u+\psi) + \gamma(u+\psi)\|\mathcal{D}u+\psi'\|]^2\mu\, ds \\
&\le \int_{-R-1}^{R+1}[\beta(u+\psi) + \gamma(u+\psi)\{\|\mathcal{D}u\| + \|\psi'\|\}]^2\mu\, ds. \qquad (5.35)
\end{aligned}$$

$\mathcal{D}u \in L_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, β and γ are continuous, and by estimate (5.10), $\|u(s)\|$ is bounded by $\|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}$ for each $s \in \mathbb{R}$. Thus the right-hand side of (5.35) is bounded independently of $u \in \Omega$, $\tau \in [0,1]$.
Also, since $f \in C^1(\mathbb{R},\mathbb{R}^N)$, for $s \in \mathbb{R}$,

$$f(u(s)+\psi(s)) = f(\psi(s)) + b(s)u(s), \qquad (5.36)$$

where

$$b(s) = \int_0^1 df[tu(s)+\psi(s)]\, dt. \qquad (5.37)$$

It follows once again from estimate (5.10) and the fact that $f \in C^1(\mathbb{R}, \mathbb{R}^N)$ that $b(s)$ is bounded independently of $s \in \mathbb{R}$ and $u \in \Omega$. Using estimate (5.10) together with the facts that $f(\psi(s)) = 0$ when $|s| \geq 1$ and $u \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, it follows that $f(u + \psi) \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $\|f(u + \psi)\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}$ is bounded independently of $u \in \Omega$.

Thus

$$\left| \int_{\mathbb{R}} \langle \tau \sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi') + f(u + \psi), v \rangle \mu \, ds \right| \leq \alpha(\Omega) \|v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \tag{5.38}$$

for some $\alpha(\Omega) \in \mathbb{R}$, dependent only on Ω. The result follows from (5.33), (5.34) and (5.38).

□

The idea of the following proof is due to J.M. Lasry, as discussed in Ekeland and Temam [16]; it stems from an original result due to Krasnoselsk'ii.

Lemma 5.2.5 *P_R is jointly continuous in τ and u (from the strong topology of $[0, 1] \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ to the strong topology of $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$).*

Proof. Let $(\tau, u) \in [0, 1] \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, $\{(\tau_n, u_n)\}_{n=1}^{\infty} \subset [0, 1] \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ be such that $(\tau_n, u_n) \to (\tau, u)$ as $n \to \infty$. We show first that as $n \to \infty$,

$$\sigma_R G(u_n + \psi, \mathcal{D}u_n + \psi')[\mathcal{D}u_n + \psi'] \to \sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi') \text{ in } L_{2,\mu}(\mathbb{R}, \mathbb{R}^N). \tag{5.39}$$

(Recall from Lemma 5.2.4, (5.35) that $\sigma_R G(u_o + \psi, \mathcal{D}u_o + \psi')[\mathcal{D}u_o + \psi'] \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ for each $u_o \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.) Choose a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ and a set $\Omega \subset \mathbb{R}$ of full measure, such that

$$\|u_{n_k} - u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} < 2^{-k}, \tag{5.40}$$

$$u_{n_k}(s) \to u(s) \text{ and } \mathcal{D}u_{n_k}(s) \to \mathcal{D}u(s) \text{ for each } s \in \Omega$$

(which is possible since $u_{n_k} \to u$ in $L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $\mathcal{D}u_{n_k} \to \mathcal{D}u$ in $L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$). Then for $s \in \Omega$,

$$\begin{aligned} &\|\sigma_R(s) G(u_{n_k}(s) + \psi(s), \mathcal{D}u_{n_k}(s) + \psi'(s))[\mathcal{D}u_{n_k}(s) + \psi'(s)] \\ &- \quad \sigma_R(s) G(u(s) + \psi(s), \mathcal{D}u(s) + \psi(s))[\mathcal{D}u(s) + \psi'(s)]\| \to 0 \end{aligned} \tag{5.41}$$

as $k \to \infty$, since G is continuous. So for each $s \in \Omega$, there exists n_k, which we label $n_{k(s)}$, such that

$$\begin{aligned}
&\|\sigma_R(s)G(u_{n_k}(s) + \psi(s), \mathcal{D}u_{n_k}(s) + \psi'(s))[\mathcal{D}u_{n_k}(s) + \psi'(s)] \\
&\quad - \quad \sigma_R G(u(s) + \psi(s), \mathcal{D}u(s) + \psi'(s))[\mathcal{D}u(s) + \psi'(s)]\| \\
&\leq \|\sigma_R(s)G(u_{n_{k(s)}}(s) + \psi(s), \mathcal{D}u_{n_{k(s)}}(s) + \psi'(s))[\mathcal{D}u_{n_{k(s)}}(s) + \psi'(s)] \\
&\quad - \quad \sigma_R(s)G(u(s) + \psi(s), \mathcal{D}u(s) + \psi'(s))[\mathcal{D}u(s) + \psi'(s)]\|
\end{aligned}$$

for all u_{n_k}.

Define functions $v, w : \mathbb{R} \to \mathbb{R}^N$ by $v(s) = u_{n_{k(s)}}(s)$, $w(s) = \mathcal{D}u_{n_{k(s)}}(s)$ for $s \in \Omega$, and $v(s) = w(s) = 0$ for $s \notin \Omega$. Then v and w are measurable (see [16]), and for $s \in \Omega$,

$$\begin{aligned}
&\|\sigma_R(s)G(u_{n_k}(s) + \psi(s), \mathcal{D}u_{n_k}(s) + \psi'(s))[\mathcal{D}u_{n_k}(s) + \psi'(s)] \\
&\quad - \quad \sigma_R G(u(s) + \psi(s), \mathcal{D}u(s) + \psi'(s))[\mathcal{D}u(s) + \psi'(s)]\| \\
&\leq \|\sigma_R(s)G(v(s) + \psi(s), w(s) + \psi'(s))[w(s) + \psi'(s)] \\
&\quad - \quad \sigma_R(s)G(u(s) + \psi(s), \mathcal{D}u(s) + \psi'(s))[\mathcal{D}u(s) + \psi'(s)]\|. \qquad (5.42)
\end{aligned}$$

Now for $s \in \mathbb{R}$,

$$\begin{aligned}
|v(s)|^2 &= |u_{n_{k(s)}}(s)|^2 \\
&\leq (|u_{n_{k(s)}}(s) - u(s)| + |u(s)|)^2 \\
&\leq 2(|u_{n_{k(s)}}(s) - u(s)|^2 + |u(s)|^2) \\
&\leq 2(|u(s)|^2 + \sum_{i=1}^{\infty} |u_{n_i}(s) - u(s)|^2). \qquad (5.43)
\end{aligned}$$

So by Levi's Monotone Convergence Theorem and (5.40),

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\sum_{i=1}^{\infty} |u_{n_i}(s) - u(s)|^2 \right) \mu(s)\, ds = \lim_{K\to\infty} \int_{\mathbb{R}} \left(\sum_{i=1}^{K} |u_{n_i}(s) - u(s)|^2 \right) \mu(s)\, ds \\
&= \lim_{K\to\infty} \sum_{i=1}^{K} \left(\int_{\mathbb{R}} |u_{n_i}(s) - u(s)|^2 \mu(s)\, ds \right) = \lim_{K\to\infty} \sum_{i=1}^{K} \|u_{n_i} - u\|^2_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \\
&\leq \lim_{K\to\infty} \sum_{i=1}^{K} \|u_{n_i} - u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \leq \lim_{K\to\infty} \sum_{i=1}^{K} 2^{-2i} = \sum_{i=1}^{\infty} 4^{-i} = \frac{1}{3}.
\end{aligned}$$

It follows from (5.43) and the fact that $u \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ that $v \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. A virtually identical argument shows that $w \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Note also that $|v(s)|$ is bounded independently of s by (5.10) since $\{u_{n_k}\}_{k=1}^{\infty}$ is bounded in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. As argued in the proof of Lemma 5.2.3, it follows that

$$\sigma_R G(v + \psi, w + \psi')[w + \psi'] \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N), \quad \text{and hence}$$

$$\sigma_R G(v + \psi, w + \psi')[w + \psi'] - \sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi') \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N).$$

So (5.41) and (5.42) together with Lebesgue's Dominated Convergence Theorem yield that

$$\sigma_R G(u_{n_k} + \psi, \mathcal{D}u_{n_k} + \psi')[\mathcal{D}u_{n_k} + \psi'] \to \sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi')$$

in $L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ as $k \to \infty$. If (5.39) did not hold, there would exist $\epsilon > 0$ and a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that for each $i \in \mathbb{N}$,

$$\|\sigma_R G(u_{n_i} + \psi, \mathcal{D}u_{n_i} + \psi')[\mathcal{D}u_{n_i} + \psi'] - \sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi')\| \geq \epsilon. \quad (5.44)$$

But the above argument would then yield a subsequence of

$$\{\sigma_R G(u_{n_i} + \psi, \mathcal{D}u_{n_i} + \psi')[\mathcal{D}u_{n_i} + \psi']\}_{i=1}^{\infty}$$

convergent to $\sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi')$, which would contradict (5.44). It follows that (5.39) holds.

A similar but slightly simpler argument shows that as $n \to \infty$,

$$f(u_n + \psi) \to f(u + \psi) \text{ in } L_{2,\mu}(\mathbb{R}, \mathbb{R}^N). \quad (5.45)$$

(Recall from the proof of Lemma 5.2.4 that $f(u_o + \psi) \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ for each $u_o \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.)

Let $v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N), n \in \mathbb{N}$. Observe first that with α_3 as in (5.33), we have that

$$\begin{aligned} \sup_{\|v\|_{W^1_{2,\mu}} = 1} \left| \int_{\mathbb{R}} \langle A[\mathcal{D}u_n - \mathcal{D}u], \mathcal{D}(v\mu) \rangle \, ds \right| &\leq \alpha_3 \|u_n - u\|_{W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)} \quad (5.46) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Now by Corollary 5.2.3, there exists $\alpha \in \mathbb{R}$, independent of n, such that $|c(u_n)| \leq \alpha$ for each n. So

$$\begin{aligned}
&\left|c(u_n)\int_{\mathbb{R}}\langle \mathcal{D}u_n + \psi', v\rangle \mu\, ds - c(u)\int_{\mathbb{R}}\langle \mathcal{D}u + \psi', v\rangle \mu\, ds\right| \\
&\leq \left\{|c(u_n)|\left|\int_{\mathbb{R}}\langle \mathcal{D}u_n - \mathcal{D}u, v\rangle \mu\, ds\right| + |c(u_n) - c(u)|\left|\int_{\mathbb{R}}\langle \mathcal{D}u + \psi', v\rangle \mu\, ds\right|\right\} \\
&\leq \Big\{|c(u_n)|\|\mathcal{D}u_n - \mathcal{D}u\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \\
&\qquad\qquad + |c(u_n) - c(u)|\|\mathcal{D}u + \psi'\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\Big\} \\
&\leq \alpha\|\mathcal{D}u_n - \mathcal{D}u\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \qquad (5.47) \\
&\qquad\qquad + |c(u_n) - c(u)|\|\mathcal{D}u + \psi'\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)},
\end{aligned}$$

and so

$$\sup_{\|v\|_{W^1_{2,\mu}}=1}\left|c(u_n)\int_{\mathbb{R}}\langle \mathcal{D}u_n + \psi', v\rangle \mu\, ds - c(u)\int_{\mathbb{R}}\langle \mathcal{D}u + \psi', v\rangle \mu\, ds\right| \to 0 \text{ as } n \to \infty \tag{5.48}$$

since c is a continuous function of u (Lemma 5.2.2). Next,

$$\begin{aligned}
&\Big|\tau_n \int_{\mathbb{R}} \sigma_R\langle G(u_n + \psi, \mathcal{D}u_n + \psi')[\mathcal{D}u_n + \psi'], v\rangle \mu\, ds \\
&\qquad\qquad - \tau\int_{\mathbb{R}} \sigma_R\langle G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi'), v\rangle \mu\, ds\Big| \\
&\leq \left|\int_{\mathbb{R}} \sigma_R\langle G(u_n + \psi, \mathcal{D}u_n + \psi')[\mathcal{D}u_n + \psi'] - G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi'), v\rangle \mu\, ds\right| \\
&\qquad\qquad + |\tau_n - \tau|\left|\int_{\mathbb{R}} \sigma_R\langle G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi'), v\rangle \mu\, ds\right| \\
&\leq \|\sigma_R G(u_n + \psi, \mathcal{D}u_n + \psi')[\mathcal{D}u_n + \psi'] \\
&\qquad - \sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi')\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \\
&\qquad + |\tau_n - \tau|\|\sigma_R G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi')\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}\|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}
\end{aligned}$$

and so by (5.39),

$$\begin{aligned}
\sup_{\|v\|_{W^1_{2,\mu}}=1}&\Big|\tau_n \int_{\mathbb{R}} \sigma_R\langle G(u_n + \psi, \mathcal{D}u_n + \psi')[\mathcal{D}u_n + \psi'], v\rangle \mu\, ds \\
&- \tau\int_{\mathbb{R}} \sigma_R\langle G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi'), v\rangle \mu\, ds\Big| \to 0 \qquad (5.49)
\end{aligned}$$

as $n \to \infty$. Finally, by (5.45),

$$\sup_{\|v\|_{W^1_{2,\mu}}=1} \left| \int_{\mathbb{R}} \langle f(u_n+\psi) - f(u+\psi), v\rangle \mu \, ds \right|$$
$$\leq \|f(u_n+\psi) - f(u+\psi)\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \to 0 \text{ as } n \to \infty. \tag{5.50}$$

Together, (5.46), (5.48), (5.49) and (5.50) show that

$$\|P_R(\tau_n, u_n) - P_R(\tau, u)\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*} \to 0 \text{ as } n \to \infty,$$

as required.

□

We now relate this operator P_R to (5.31).

Lemma 5.2.6 *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $\tau \in [0,1]$. Then u is a solution of $P_R(\tau, u) = 0$ if and only if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and u satisfies (5.31).*

Proof. Suppose first that $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, $\tau \in [0,1]$ are such that $P_R(\tau, u) = 0$. Then for each $v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,

$$\int_{\mathbb{R}} \langle A\mathcal{D}u, \mathcal{D}(v\mu)\rangle \, ds = \tag{5.51}$$
$$\int_{\mathbb{R}} \langle A\psi'' + c(u)[\mathcal{D}u + \psi'] + \tau\sigma_R G(u+\psi, \mathcal{D}u+\psi')(\mathcal{D}u+\psi') + f(u+\psi), v\rangle \mu \, ds.$$

So for $\theta \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$, $v = \dfrac{\theta}{\mu}$,

$$\int_{\mathbb{R}} \langle A\mathcal{D}u, \theta'\rangle \, ds = \tag{5.52}$$
$$\int_{\mathbb{R}} \langle A\psi'' + c(u)[\mathcal{D}u + \psi'] + \tau\sigma_R G(u+\psi, \mathcal{D}u+\psi')(\mathcal{D}u+\psi') + f(u+\psi), \theta\rangle \, ds,$$

since $\dfrac{\theta}{\mu} \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Hence $\mathcal{D}u$ has a weak derivative,

$$-A^{-1}[A\psi'' + c(u)[\mathcal{D}u + \psi'] + \tau\sigma_R G(u+\psi, \mathcal{D}u+\psi')(\mathcal{D}u+\psi') + f(u+\psi)],$$

which can be shown to be in $L_2(\mathcal{I}, \mathbb{R}^N)$ for bounded $\mathcal{I} \in \mathbb{R}$ using estimates (5.35) and (5.36). So $u \in W^2_2(\mathcal{I}, \mathbb{R}^N)$ for bounded $\mathcal{I} \subset \mathbb{R}$. The Sobolev Embedding

Theorem thus yields that $u \in C^1(\mathbb{R}, \mathbb{R}^N)$. Moreover, u' ($= \mathcal{D}u$ almost everywhere) is absolutely continuous since it has a locally integrable weak derivative, so the classical derivative of u' exists almost everywhere and is equal to this weak derivative. Hence by the Fundamental Theorem of Calculus,

$$\begin{aligned} u'(s) = u'(s_0) - \int_{s_0}^{s} A^{-1}[A\psi''(t) + c(u)[u'(t) + \psi'(t)] \\ +\tau\sigma_R(t)G(u(t) + \psi(t), u'(t) + \psi'(t))[u'(t) + \psi'(t)] + f(u(t) + \psi(t))]\, dt \end{aligned} \tag{5.53}$$

for each $s, s_0 \in \mathbb{R}$. Since the integrand on the right-hand side of (5.53) is continuous, $u' \in C^1(\mathbb{R}, \mathbb{R}^N)$ and

$$\begin{aligned} u''(s) = -A^{-1}[A\psi''(s) + c(u)[u'(s) + \psi'(s)] \\ +\tau\sigma_R(s)G(u(s) + \psi(s), u'(s) + \psi'(s))[u'(s) + \psi'(s)] + f(u(s) + \psi(s))] \end{aligned} \tag{5.54}$$

for each $s \in \mathbb{R}$. So $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and satisfies (5.31), as required.

Conversely, suppose that $\tau \in [0, 1]$ and $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (5.31). Let $v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $\{\theta_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}, \mathbb{R}^N)$ converge to v in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Clearly, $(P_R(\tau, u))(\theta_n) = 0$ for each n, so since

$$(P_R(\tau, u))(v - \theta_n) \to 0 \ \text{ as } n \to \infty,$$

it follows that $P_R(\tau, u)(v) = 0$.

□

Following [42], we will construct an operator $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ that satisfies the following.

Theorem 5.2.7 *There exists a bounded linear positive-definite self-adjoint operator $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and a function $\theta : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \times [0, 1]$, such that for $\tau \in [0, 1]$ and $u, u_0 \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,*

$$(P_R(\tau, u))(S_\mu(u - u_0)) \geq \|u - u_0\|^2_{W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)} + \theta(u, u_0, \tau), \tag{5.55}$$

and $\theta(u_n, u_0, \tau) \to 0$ uniformly for $\tau \in [0, 1]$ as $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.

We will return to the proof of this result in a moment. First note the following.

Lemma 5.2.8 *If H is a Hilbert space and $S : H \to H$ is a bounded linear self-adjoint positive-definite operator, then $S^* : H^* \to H^*$ is bounded linear and injective.*

[Recall that S^* is the conjugate of S, defined by $(S^*f)(u) = f(Su)$ for each $u \in H$, $f \in H^*$.]

Proof. Linearity and boundedness of S^* clearly follow from the corresponding properties of S. We prove injectivity. Let $f \in H^*$ and suppose that $S^*f = 0$. Then for every $u \in H$, $f(Su) = (S^*f)(u) = 0$. By the Riesz Representation Theorem, there exists some $v \in H$ such that $f(w) = \langle w, v \rangle_H$ for each $w \in H$; in particular, $\langle Sv, v \rangle_H = f(Sv) = 0$. Since S is positive-definite, $v = 0$, and hence $f = 0$. The result follows.

□

This, together with Lemma 5.2.6, yields that for $\tau \in [0,1]$ and $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,

$$S^*_\mu P_R(\tau, u) = 0 \Leftrightarrow u \in C^2(\mathbb{R}, \mathbb{R}^N) \text{ and } u \text{ satisfies (5.31).} \tag{5.56}$$

Recall the discussion of degree theory for $(S)_+$ operators in section 5.1.2. Lemmas 5.2.4, 5.2.5 and 5.2.8 show that $S^*_\mu P_R$ is jointly continuous in τ and u, and maps each bounded subset of $[0,1] \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ into a bounded subset of $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$. Moreover, (5.55) can be rewritten as

$$(S^*_\mu P_R(\tau, u))(u - u_0) \geq \|u - u_0\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta(u, u_0, \tau). \tag{5.57}$$

So for each $\tau \in [0,1]$, $S^*_\mu P_R(\tau, \cdot)$ satisfies condition (5.11), and hence is of class $(S)_+$. Also, since $\theta(u, u_0, \tau) \to 0$ *uniformly for* $\tau \in [0,1]$ as $u \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, the operator $S^*_\mu P_R : [0,1] \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ satisfies condition (5.14) required for an admissible $(S)_+$ homotopy.

So if $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ is open, bounded and such that

$$0 \notin S^*_\mu P_R([0,1] \times \partial\Omega), \tag{5.58}$$

then $S_\mu^* P_R(0,\cdot)$ and $S_\mu^* P_R(1,\cdot)$ are $(S)_+$ homotopic relative to Ω. Hence

$$deg_{(S)+}(S_\mu^* P_R(1,\cdot),\Omega,0) = deg_{(S)+}(S_\mu^* P_R(0,\cdot),\Omega,0). \tag{5.59}$$

Further, if $0 \notin S_\mu^* P_R(\tau,\cdot)(\partial\Omega)$ and $deg_{(S)+}(S_\mu^* P_R(\tau,\cdot),\Omega,0) \neq 0$, then there exists $u \in \Omega$ such that $S_\mu^* P_R(\tau,u) = 0$, so by (5.56), $u \in C^2(\mathbb{R},\mathbb{R}^N)$ satisfies (5.31).

Chapter 6 is devoted to proving *a priori* bounds for solutions of $S_\mu^* P_R(\tau,u) = 0$ where $u+\psi$ is monotone (as defined in Chapter 1); by virtue of (5.56), this is equivalent to bounding monotone solutions of (5.31). The rest of the current chapter is concerned with proving Theorem 5.2.7. Our approach here extends that of [42]. We begin with some preliminary estimates for linear differential operators with constant coefficients. The proof given here corrects an oversight in [42], where they omit to take real parts when it is clearly necessary to do so.

Lemma 5.2.9 *Let $A \in M^{N\times N}$ be a positive-definite diagonal matrix and let $B_0 \in P^{N\times N}$ have $\mu_{PF}(B_0) < 0$. Then there exists a bounded linear positive-definite self-adjoint operator $T_0 : L_2(\mathbb{R},\mathbb{R}^N) \to L_2(\mathbb{R},\mathbb{R}^N)$ such that $T_0|_{W_2^1(\mathbb{R},\mathbb{R}^N)} : W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ is also bounded, and*

$$(L_0u)(T_0u) = \|u\|^2_{W_2^1(\mathbb{R},\mathbb{R}^N)} \text{ for each } u \in W_2^1(\mathbb{R},\mathbb{R}^N), \tag{5.60}$$

where for $u,v \in W_2^1(\mathbb{R},\mathbb{R}^N)$, $L_0 \in W_2^1(\mathbb{R},\mathbb{R}^N)^*$ is defined by

$$(L_0u)(v) = \int_{\mathbb{R}} [\langle A(\mathcal{D}u)(s),(\mathcal{D}v)(s)\rangle - \langle B_0u(s),v(s)\rangle]\,ds. \tag{5.61}$$

Proof. For $\xi \in \mathbb{R}$, let $C(\xi)$ be the real matrix

$$C(\xi) := \frac{-\xi^2 A + B_0}{1+\xi^2}. \tag{5.62}$$

and define

$$\mathcal{C} := \{C(\xi) : \xi \in \mathbb{R}\} \cup \{-A\}. \tag{5.63}$$

Recall Theorems 2.2.1 and 2.2.2. It follows that $\mu_{PF}(C(\xi)) < 0$ for each $\xi \in \mathbb{R}$, since $\mu_{PF}(B_0) < 0$ and A is a positive-definite diagonal matrix. Moreover, the continuous dependence of the eigenvalues of $C(\xi)$ on ξ and the fact that the eigenvalues tend to those of $-A$ and B_0 as $|\xi| \to \infty$ and $\xi \to 0$ respectively together imply the existence of an open bounded set $\Omega \in \mathbb{C}$ and of $\omega > 0$ such that Re $z \leq -\omega$ for each $z \in \Omega$, and for each $M \in \mathcal{C}$, all eigenvalues of M are contained in Ω. Without loss of generality, suppose that $\partial\Omega$ is a smooth closed simple curve with a positive anticlockwise orientation.

As in [42], let $R(\xi)$ denote the real, symmetric positive-definite matrix

$$R(\xi) := 2\int_0^\infty e^{sC^*(\xi)}e^{sC(\xi)}\,ds, \tag{5.64}$$

where $C^*(\xi)$ is the matrix adjoint of $C(\xi)$. To show that $R(\xi)$ is well-defined, note first that by Cauchy's Integral Formula,

$$e^{sC(\xi)} = \frac{1}{2\pi i}\int_{\partial\Omega} e^{s\lambda}(\lambda I - C(\xi))^{-1}\,d\lambda. \tag{5.65}$$

The set $\mathcal{C}$ is compact in $M^{N\times N}$, from which it follows that $\sup\limits_{\lambda\in\Omega}\|(\lambda I - C(\xi))^{-1}\|$ is bounded independently of $\xi \in \mathbb{R}$. Whence there exists a constant $K \in \mathbb{R}$, independent of ξ, such that

$$\|e^{sC(\xi)}\| \leq Ke^{-\omega s}, \tag{5.66}$$

where here $\|\cdot\|$ denotes a norm on $M^{N\times N}$; similarly,

$$\|e^{sC^*(\xi)}\| \leq Ke^{-\omega s}. \tag{5.67}$$

In fact, (5.66) and (5.67) show that $\|R(\xi)\|$ is bounded independently of $\xi \in \mathbb{R}$. Now for any $p \in \mathbb{R}^N$,

$$\begin{aligned}\langle C(\xi)p, R(\xi)p\rangle &= \int_0^\infty \left\{\langle e^{sC(\xi)}C(\xi)p, e^{sC(\xi)}p\rangle + \langle e^{sC(\xi)}p, e^{sC(\xi)}C(\xi)p\rangle\right\} ds \\ &= \int_0^\infty \frac{d}{ds}\langle e^{sC(\xi)}p, e^{sC(\xi)}p\rangle\,ds \\ &= -\langle p,p\rangle, \quad \text{by (5.66)}. \end{aligned} \tag{5.68}$$

Thus for any $q \in \mathbb{C}^N$,

$$\mathrm{Re}\,\langle C(\xi)q, R(\xi)q\rangle_{\mathbb{C}^N} = -\langle q,q\rangle_{\mathbb{C}^N}. \tag{5.69}$$

Next define $\Psi : L_2(\mathbb{R}, \mathbb{R}^N) \to \tilde{L}_2(\mathbb{R}, \mathbb{C}^N)$ by

$$\widehat{\Psi u}(\xi) = R(\xi)\hat{u}(\xi), \quad \xi \in \mathbb{R}, u \in L_2(\mathbb{R}, \mathbb{R}^N), \tag{5.70}$$

where for each $w \in \tilde{L}_2(\mathbb{R}, \mathbb{C}^N)$, $\hat{w}$ denotes the Fourier transform of w (here, ξ denotes the transform variable, as usual). Ψ is well-defined because $R(\xi)$ is bounded independently of ξ.

Let $T_0 : L_2(\mathbb{R}, \mathbb{R}^N) \to L_2(\mathbb{R}, \mathbb{R}^N)$ be defined by

$$T_0 u = \mathrm{Re}\,(\Psi u) \;\text{ for } u \in L_2(\mathbb{R}, \mathbb{R}^N). \tag{5.71}$$

Clearly, T_0 is a bounded linear operator. For each $u, v \in L_2(\mathbb{R}, \mathbb{R}^N)$, Parseval's formula and the symmetry of $R(\xi)$ yield that

$$\begin{aligned}
&\langle T_0 u, v\rangle_{L_2(\mathbb{R},\mathbb{R}^N)} = \mathrm{Re}\,\langle \Psi u, v\rangle_{\tilde{L}_2(\mathbb{R},\mathbb{C}^N)} \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle \widehat{\Psi u}(\xi), \hat{v}(\xi)\rangle_{\mathbb{C}^N}\, d\xi = \mathrm{Re} \int_{\mathbb{R}} \langle R(\xi)\hat{u}(\xi), \hat{v}(\xi)\rangle_{\mathbb{C}^N}\, d\xi \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle \hat{u}(\xi), R(\xi)\hat{v}(\xi)\rangle_{\mathbb{C}^N}\, d\xi = \mathrm{Re} \int_{\mathbb{R}} \overline{\langle R(\xi)\hat{v}(\xi), \hat{u}(\xi)\rangle}_{\mathbb{C}^N}\, d\xi \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle R(\xi)\hat{v}(\xi), \hat{u}(\xi)\rangle_{\mathbb{C}^N}\, d\xi = \langle T_0 v, u\rangle_{L_2(\mathbb{R},\mathbb{R}^N)},
\end{aligned}$$

and hence T_0 is self-adjoint.

Furthermore, for $u \in L_2(\mathbb{R}, \mathbb{R}^N)$,

$$\begin{aligned}
\langle T_0 u, u\rangle_{L_2(\mathbb{R},\mathbb{R}^N)} &= \mathrm{Re} \int_{\mathbb{R}} \langle R(\xi)\hat{u}(\xi), \hat{u}(\xi)\rangle_{\mathbb{C}^N}\, d\xi \\
&> 0 \;\text{ if } u \neq 0,
\end{aligned}$$

because $R(\xi)$ is positive-definite for each ξ. So T_0 is positive-definite.

Now let $u \in W_2^1(\mathbb{R}, \mathbb{R}^N)$. Then for $\theta \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$,

$$\begin{aligned}
&\int_{\mathbb{R}} \langle T_0 u, \theta'\rangle\, ds = \mathrm{Re} \int_{\mathbb{R}} \langle \Psi u(s), \theta'(s)\rangle_{\mathbb{C}^N}\, ds \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle \widehat{\Psi u}(\xi), \widehat{\theta'}(\xi)\rangle_{\mathbb{C}^N}\, d\xi = \mathrm{Re} \int_{\mathbb{R}} \langle i\xi R(\xi)\hat{u}(\xi), \hat{\theta}(\xi)\rangle_{\mathbb{C}^N}\, d\xi \\
&= -\mathrm{Re} \int_{\mathbb{R}} \langle R(\xi)\widehat{\mathcal{D}u}(\xi), \hat{\theta}(\xi)\rangle_{\mathbb{C}^N}\, d\xi = -\mathrm{Re} \int_{\mathbb{R}} \langle \widehat{\Psi(\mathcal{D}u)}(\xi), \hat{\theta}(\xi)\rangle_{\mathbb{C}^N}\, d\xi \\
&= -\mathrm{Re} \int_{\mathbb{R}} \langle \Psi(\mathcal{D}u)(s), \theta(s)\rangle_{\mathbb{C}^N}\, ds = -\int_{\mathbb{R}} \langle T_0(\mathcal{D}u), \theta\rangle\, ds.
\end{aligned}$$

Therefore $T_0 u$ has a weak derivative,

$$\mathcal{D}(T_0 u) = T_0(\mathcal{D}u) \in L_2(\mathbb{R}, \mathbb{R}^N). \tag{5.72}$$

So $T_0 u \in W_2^1(\mathbb{R}, \mathbb{R}^N)$. That $T_0|_{W_2^1(\mathbb{R},\mathbb{R}^N)}$ is bounded in $W_2^1(\mathbb{R}, \mathbb{R}^N)$ follows from the boundedness of T_0 in $L_2(\mathbb{R}, \mathbb{R}^N)$ and (5.72).
Finally, (5.61) and Parseval's formula yield that for $u, v \in W_2^1(\mathbb{R}, \mathbb{R}^N)$,

$$\begin{aligned}
(L_0 u)(v) &= \int_{\mathbb{R}} [\langle A(\mathcal{D}u)(s), (\mathcal{D}v)(s)\rangle - \langle B_0 u(s), v(s)\rangle]\, ds \\
&= \int_{\mathbb{R}} [\langle \xi^2 A\hat{u}(\xi), \hat{v}(\xi)\rangle_{\mathbb{C}^N} - \langle B_0 \hat{u}(\xi), \hat{v}(\xi)\rangle_{\mathbb{C}^N}]\, d\xi \\
&= -\int_{\mathbb{R}} (1+\xi^2)\langle C(\xi)\hat{u}(\xi), \hat{v}(\xi)\rangle_{\mathbb{C}^N}\, d\xi.
\end{aligned}$$

Hence for $u \in W_2^1(\mathbb{R}, \mathbb{R}^N)$,

$$\begin{aligned}
(L_0 u)(T_0 u) &= \mathrm{Re}\,(L_0 u)(\Psi u) \\
&= -\mathrm{Re} \int_{\mathbb{R}} (1+\xi^2)\langle C(\xi)\hat{u}(\xi), \widehat{\Psi u}(\xi)\rangle_{\mathbb{C}^N}\, d\xi \\
&= -\mathrm{Re} \int_{\mathbb{R}} (1+\xi^2)\langle C(\xi)\hat{u}(\xi), R(\xi)\hat{u}(\xi)\rangle_{\mathbb{C}^N}\, d\xi \\
&= \int_{\mathbb{R}} (1+\xi^2)\langle \hat{u}(\xi), \hat{u}(\xi)\rangle\, d\xi \qquad \text{by (5.69)}, \\
&= \|u\|^2_{W_2^1(\mathbb{R},\mathbb{R}^N)},
\end{aligned}$$

as required.

□

Remark. It follows from (5.72) and the self-adjointness of $T_0 : L_2(\mathbb{R}, \mathbb{R}^N) \to L_2(\mathbb{R}, \mathbb{R}^N)$ that $T_0 \big|_{W_2^1(\mathbb{R},\mathbb{R}^N)}$ is self adjoint with respect to the inner product in $W_2^1(\mathbb{R}, \mathbb{R}^N)$.
To proceed, the following technical lemma is useful.

Lemma 5.2.10 *Let* $\{f_n\} \subset L_2(\mathbb{R}, \mathbb{R}^N)$ *and* $\{g_n\} \subset W_2^1(\mathbb{R}, \mathbb{R}^N)$ *be bounded sequences of vector-valued functions with* $g_n \rightharpoonup 0$ *in* $W_2^1(\mathbb{R}, \mathbb{R}^N)$ *as* $n \to \infty$. *Let* $\vartheta : \mathbb{R} \to \mathbb{R}$ *belong to scalar-valued* $L_2(\mathbb{R}, \mathbb{R})$. *Then*

$$\int_{\mathbb{R}} \vartheta(s)\langle f_n(s), g_n(s)\rangle\, ds \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Since $\{g_n\}$ is bounded in $W_2^1(\mathbb{R}, \mathbb{R}^N)$, it is bounded in $L_\infty(\mathbb{R}, \mathbb{R}^N)$; say $\|g_n(s)\| \leq M$ almost everywhere. Since ϑ is square integrable, given $\epsilon > 0$, there exists $\kappa > 0$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned}\left|\int_{|s|>\kappa} \vartheta(s)\langle f_n(s), g_n(s)\rangle \, ds\right| &\leq \int_{|s|>\kappa} |\vartheta(s)|\|f_n(s)\|\|g_n(s)\| \, ds \\ &\leq M\|f_n\|_{L_2(\mathbb{R},\mathbb{R}^N)} \left(\int_{|s|>\kappa} |\vartheta(s)|^2 \, ds\right)^{\frac{1}{2}} < \frac{\epsilon}{2}.\end{aligned}$$

Moreover, by the Sobolev Embedding Theorem, $g_n(s) \to 0$ uniformly for $s \in [-\kappa, \kappa]$ as $n \to \infty$, so there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\left|\int_{|s|\leq\kappa} \vartheta(s)\langle f_n(s), g_n(s)\rangle \, ds\right| < \frac{\epsilon}{2}.$$

The result follows.

□

Now we can develop Lemma 5.2.9 to a stronger statement about operators with variable coefficients.

Lemma 5.2.11 *Let A be a positive-definite diagonal matrix and B_1 and B_2 satisfy the conditions on B_0 in Lemma 5.2.9. Let the matrix B be given by*

$$B(s) = \phi_1(s)B_1 + \phi_2(s)B_2, \quad s \in \mathbb{R}, \tag{5.73}$$

where $\phi_1, \phi_2 \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfy $0 \leq \phi_1(s), \phi_2(s) \leq 1$ and $\phi_1(s) + \phi_2(s) = 1$ for each $s \in \mathbb{R}$, and $\phi_1(s) = 0$ when $s > 1$, $\phi_2(s) = 0$ when $s < -1$. Then there exists a bounded linear positive-definite self-adjoint operator $S : W_2^1(\mathbb{R}, \mathbb{R}^N) \to W_2^1(\mathbb{R}, \mathbb{R}^N)$ such that for $u \in W_2^1(\mathbb{R}, \mathbb{R}^N)$,

$$(Lu)(Su) \geq \|u\|_{W_2^1(\mathbb{R},\mathbb{R}^N)} + \theta(u), \tag{5.74}$$

where $\theta : W_2^1(\mathbb{R}, \mathbb{R}^N) \to \mathbb{R}$ is such that $\theta(u) \to 0$ as $u \rightharpoonup 0$ in $W_2^1(\mathbb{R}, \mathbb{R}^N)$. Here $Lu \in W_2^1(\mathbb{R}, \mathbb{R}^N)^*$ is defined by

$$(Lu)(v) = \int_{\mathbb{R}} [\langle A(\mathcal{D}u)(s), (\mathcal{D}v)(s)\rangle - \langle B(s)u(s), v(s)\rangle] \, ds, \tag{5.75}$$

for $v \in W_2^1(\mathbb{R}, \mathbb{R}^N)$.

Proof. Define the operator $T : W_2^1(\mathbb{R}, \mathbb{R}^N) \to W_2^1(\mathbb{R}, \mathbb{R}^N)$ by

$$(Tu)(s) = \sum_{i=1}^{2} \phi_i(s) T_i(\phi_i u)(s), \quad u \in W_2^1(\mathbb{R}, \mathbb{R}^N), \quad s \in \mathbb{R}, \tag{5.76}$$

where $T_i : W_2^1(\mathbb{R}, \mathbb{R}^N) \to W_2^1(\mathbb{R}, \mathbb{R}^N)$ is defined using B_i as T_0 was defined using B_0 in Lemma 5.2.9. Now

$$\begin{aligned}
\int_{\mathbb{R}} \langle A(\mathcal{D}u)(s), (\mathcal{D}(Tu))(s)\rangle \, ds &= \int_{\mathbb{R}} \langle A(\mathcal{D}u)(s), \sum_{i=1}^{2} \phi_i'(s) T_i(\phi_i u)(s)\rangle \, ds \\
&\quad + \int_{\mathbb{R}} \langle A(\mathcal{D}u)(s), \sum_{i=1}^{2} \phi_i(s) \mathcal{D}(T_i(\phi_i u))(s)\rangle \, ds \\
&= \sum_{i=1}^{2} \left\{ \begin{array}{l} \int_{\mathbb{R}} \langle A\mathcal{D}(\phi_i u)(s), \mathcal{D}(T_i(\phi_i u))(s)\rangle \, ds \\ + \int_{\mathbb{R}} \phi_i'(s) [\langle A(\mathcal{D}u)(s), (T_i(\phi_i u))(s)\rangle \\ \qquad - \langle Au(s), \mathcal{D}(T_i(\phi_i u))(s)\rangle] \, ds \end{array} \right\} \\
&= \sum_{i=1}^{2} \left\{ \int_{\mathbb{R}} \langle A\mathcal{D}(\phi_i u)(s), \mathcal{D}(T_i(\phi_i u))(s)\rangle \, ds \right\} + \theta_1(u),
\end{aligned} \tag{5.77}$$

say. The second equality can be seen by differentiating the product $\phi_i u$ in the first integral on the right-hand side. If $u_n \rightharpoonup 0$ in $W_2^1(\mathbb{R}, \mathbb{R}^N)$, then $T_i(\phi_i u_n) \rightharpoonup 0$ in $W_2^1(\mathbb{R}, \mathbb{R}^N)$, and $\|u_n(s)\|, \|T_i(\phi_i u_n)(s)\|$ each tend to zero uniformly on compact subsets of $\mathbb{R}$. Moreover, $\{\|\mathcal{D}u_n\|_{L_2(\mathbb{R},\mathbb{R}^N)}\}$ and $\{\|\mathcal{D}(T_i(\phi_i u_n))\|_{L_2(\mathbb{R},\mathbb{R}^N)}\}$ are each bounded independently of n since $\{u_n\}$ is bounded in $W_2^1(\mathbb{R}, \mathbb{R}^N)$. So since ϕ_i' has compact support, $\theta_1(u_n) \to 0$ as $u_n \rightharpoonup 0$. Also,

$$\int_{\mathbb{R}} \langle B(s)u(s), (Tu)(s)\rangle \, ds = \int_{\mathbb{R}} \langle \sum_{i=1}^{2} \phi_i B_i u, \sum_{j=1}^{2} \phi_j (T_j(\phi_j u))\rangle \, ds$$

$$= \sum_{i=1}^{2}\left\{\int_{\mathbb{R}}\langle B_i\phi_i u, (T_i(\phi_i u))\rangle\, ds\right\} + \theta_2(u) \quad (5.78)$$

where

$$\theta_2(u) = \sum_{i\neq j}\int_{-1}^{1}\langle \phi_i B_i u, \phi_j(T_j(\phi_j u))\rangle\, ds + \sum_{i=1}^{2}\int_{-1}^{1}(\phi_i^2 - \phi_i)\langle B_i u, T_i(\phi_i u)\rangle\, ds, \quad (5.79)$$

since if $s \notin [-1,1]$, then $\phi_1(s)\phi_2(s) = 0$ and $\phi_i(s)^2 = \phi_i(s), i = 1,2$. It follows from (5.79) that $\theta_2(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W_2^1(\mathbb{R},\mathbb{R}^N)$. Thus (5.77) and (5.78) together with Lemma 5.2.9 yield that

$$\begin{aligned}(Lu)(Tu) &= \sum_{i=1}^{2}\left\{\int_{\mathbb{R}}[\langle A\mathcal{D}(\phi_i u), \mathcal{D}(T_i(\phi_i u))\rangle - \langle B_i\phi_i u, (T_i(\phi_i u))\rangle]\, ds\right\} + \theta_1(u) + \theta_2(u) \\ &= \sum_{i=1}^{2}\|\phi_i u\|^2_{W_2^1(\mathbb{R},\mathbb{R}^N)} + \theta_1(u) + \theta_2(u) \geq \frac{1}{2}\|u\|^2_{W_2^1(\mathbb{R},\mathbb{R}^N)} + \theta(u)\end{aligned} \quad (5.80)$$

where $\theta(u) := \theta_1(u) + \theta_2(u)$, so that $\theta(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W_2^1(\mathbb{R},\mathbb{R}^N)$. The last inequality follows from the fact that $u = \sum_{i=1}^{2}\phi_i u$, so

$$\|u\|^2_{W_2^1(\mathbb{R},\mathbb{R}^N)} \leq \left(\sum_{i=1}^{2}\|\phi_i u\|_{W_2^1(\mathbb{R},\mathbb{R}^N)}\right)^2 \leq 2\sum_{i=1}^{2}\|\phi_i u\|^2_{W_2^1(\mathbb{R},\mathbb{R}^N)}.$$

In general, $T : W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ is not self-adjoint. We show next that there exists a bounded linear self-adjoint positive-definite operator $S : W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ and a compact operator $K : W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ such that

$$T = \frac{1}{2}S + K. \quad (5.81)$$

For $u, v \in W_2^1(\mathbb{R},\mathbb{R}^N)$, define

$$\begin{aligned}\Phi_0(u,v) &:= \langle Tu, v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} - \langle u, Tv\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} \quad (5.82)\\ &= \sum_{i=1}^{2}\left\{\begin{array}{l}\int_{\mathbb{R}}\phi_i'\langle T_i(\phi_i u), \mathcal{D}v\rangle + \langle u, T_i(\phi_i \mathcal{D}v)\rangle \\ -\langle \mathcal{D}u, T_i(\phi_i v)\rangle - \langle T_i(\phi_i\mathcal{D}u), v\rangle]\, ds\end{array}\right\}\end{aligned} \quad (5.83)$$

using (5.76), (5.72) and the fact that T_i is self-adjoint in $L_2(\mathbb{R},\mathbb{R}^N)$ for each i.

Clearly, $\Phi_0 : W_2^1(\mathbb{R},\mathbb{R}^N) \times W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ is a bounded bilinear functional on $W_2^1(\mathbb{R},\mathbb{R}^N)$. Hence the Riesz Representation Theorem yields the existence of a bounded linear operator $K_1 : W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ such that for each $u, v \in W_2^1(\mathbb{R},\mathbb{R}^N)$,

$$\Phi_0(u,v) = \langle K_1 u, v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)}. \tag{5.84}$$

Let $\{u_n\}_{n=1}^{\infty} \subset W_2^1(\mathbb{R},\mathbb{R}^N)$ be such that $u_n \rightharpoonup 0$ in $W_2^1(\mathbb{R},\mathbb{R}^N)$ as $n \to \infty$, and let $y_n = K_1 u_n$ for each n. Then

$$\|y_n\|^2_{W_2^1(\mathbb{R},\mathbb{R}^N)} = \langle y_n, K_1 u_n\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} = \Phi_0(y_n, u_n). \tag{5.85}$$

Now $y_n \rightharpoonup 0, T_i(\phi_i u_n) \rightharpoonup 0$ and $T_i(\phi_i y_n) \rightharpoonup 0$ in $W_2^1(\mathbb{R},\mathbb{R}^N)$. Moreover, $\|\mathcal{D}u_n\|_{L_2(\mathbb{R},\mathbb{R}^N)}$, $\|\mathcal{D}y_n\|_{L_2(\mathbb{R},\mathbb{R}^N)}$, $\|T_i(\phi_i \mathcal{D}u_n)\|_{L_2(\mathbb{R},\mathbb{R}^N)}$ and $\|T_i(\phi_i \mathcal{D}y_n)\|_{L_2(\mathbb{R},\mathbb{R}^N)}$ are all bounded independently of n. So since ϕ_i' has compact support, it follows from (5.83) and Lemma 5.2.10 that $\Phi_0(y_n, u_n) \to 0$ as $n \to \infty$. Thus by (5.85), $K_1 u_n \to 0$ in $W_2^1(\mathbb{R},\mathbb{R}^N)$, and hence K_1 is compact (see Yosida [47], page 126).

Now for $u, v \in W_2^1(\mathbb{R},\mathbb{R}^N)$,

$$\langle Tu, v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} + \langle u, Tv\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} = \Phi_1(u,v) + \Phi_2(u,v), \tag{5.86}$$

where

$$\begin{aligned}
\Phi_1(u,v) &= \sum_{i=1}^{2} \int_{\mathbb{R}} [\langle \mathcal{D}(T_i(\phi_i u)), \mathcal{D}(\phi_i v)\rangle + \langle \mathcal{D}(\phi_i u), \mathcal{D}(T_i(\phi_i v))\rangle \\
&\qquad\qquad + 2\langle (T_i(\phi_i u)), (\phi_i v)\rangle]\, ds, \qquad (5.87)\\
\Phi_2(u,v) &= \sum_{i=1}^{2} \int_{\mathbb{R}} [\langle \phi_i'(T_i(\phi_i u)), \mathcal{D}v\rangle - \langle \mathcal{D}(T_i(\phi_i u)), \phi_i' v\rangle \\
&\quad + \langle \mathcal{D}u, \phi_i'(T_i(\phi_i v))\rangle - \langle \phi_i' u, \mathcal{D}(T_i(\phi_i v))\rangle]\, ds.
\end{aligned}$$
$$\tag{5.88}$$

Again, Φ_1 and Φ_2 are bounded bilinear functionals on $W_2^1(\mathbb{R},\mathbb{R}^N)$, so as above, there exist bounded linear operators $S, K_2 : W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ such that for $u, v \in W_2^1(\mathbb{R}, \mathbb{R}^N)$,

$$\Phi_1(u,v) = \langle Su, v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} \text{ and } \Phi_2(u,v) = \langle K_2 u, v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)}. \tag{5.89}$$

Now S is self-adjoint since Φ_1 is symmetric in u and v, and is positive-definite since each T_i is self-adjoint and positive-definite in $L_2(\mathbb{R}, \mathbb{R}^N)$, and (5.72) holds. As in the analysis of the operator Φ_0 above, it follows from Lemma 5.2.10 that the operator K_2 is compact.
(5.82) and (5.86) together give that for $u, v \in W_2^1(\mathbb{R}, \mathbb{R}^N)$,

$$\begin{aligned} 2\langle Tu, v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} &= \Phi_0(u,v) + \Phi_1(u,v) + \Phi_2(u,v) \\ &= \langle (S + K_1 + K_2)u, v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)}. \end{aligned}$$

Hence putting $K = \frac{1}{2}(K_1 + K_2)$, we find that S and K satisfy (5.81). The lemma follows from (5.80) and (5.81).

□

In final preparation for the proof of Theorem 5.2.7, we now obtain estimates in the weighted space $W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$. This is facilitated by the following correspondence between operators in the weighted and unweighted spaces.

Lemma 5.2.12 *Let $\mathcal{M}$: $W_2^1(\mathbb{R}, \mathbb{R}^N) \to W_2^1(\mathbb{R}, \mathbb{R}^N)$ be a bounded linear operator, and define $\mathcal{M}_\natural : W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N) \to W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$ by*

$$\langle u, \mathcal{M}_\natural v\rangle_{W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)} = \langle \omega u, \mathcal{M}\omega v\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} \tag{5.90}$$

for each $u, v \in W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$, where $\omega = \sqrt{\mu}$, as defined in (5.7). Then

(i) *$\mathcal{M}_\natural$ is a bounded linear operator;*

(ii) *If $\mathcal{M}$ is compact in $W_2^1(\mathbb{R}, \mathbb{R}^N)$, then $\mathcal{M}_\natural$ is compact in $W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$;*

(iii) *If $\mathcal{M}$ is symmetric and positive-definite in $W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$, then $\mathcal{M}_\natural$ is symmetric and positive-definite in $W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$;*

(iv) *$\mathcal{M}_\natural - \omega^{-1}\mathcal{M}\omega$ is a compact linear operator in $W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$.*

Proof. The linearity of $\mathcal{M}_\natural$ and (iii) follow directly from definition (5.90) and the properties of $\mathcal{M}$. Recall from Lemma 5.1.1 that the operator of multiplication by ω is a bounded linear operator from $W_2^1(\mathbb{R},\mathbb{R}^N)$ to $W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)$, and the operator of multiplication by ω^{-1} is a bounded linear operator from $W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)$ to $W_2^1(\mathbb{R},\mathbb{R}^N)$. Thus boundedness and compactness of $\mathcal{M}_\natural$ follow from boundedness and compactness of $\mathcal{M}$ respectively. It remains to verify property (iv).

Let $\tilde{\mathcal{M}} : W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N) \to W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)$ be defined by $\tilde{\mathcal{M}} = \omega^{-1}\mathcal{M}\omega$. By the remark earlier in this proof, $\tilde{\mathcal{M}}$ is a bounded linear operator. Now for $u, v \in W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)$,

$$\langle u, \tilde{\mathcal{M}}v\rangle_{W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)} = \int_{\mathbb{R}} [\langle \mathcal{D}u(s), \mathcal{D}(\tilde{\mathcal{M}}v)(s)\rangle + \langle u(s), \tilde{\mathcal{M}}v(s)\rangle]\mu(s)\, ds.$$

Let $y = \omega u$ and $z = \omega v$. Then $y, z \in W_2^1(\mathbb{R},\mathbb{R}^N)$, and

$$\langle u, \tilde{\mathcal{M}}v\rangle_{W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)} = \langle y, \mathcal{M}z\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)} + \Phi(y,z), \tag{5.91}$$

where

$$\begin{aligned}\Phi(y,z) \quad = \quad & \int_{\mathbb{R}} [\langle(\omega^{-1})'\omega y, , \mathcal{D}(\mathcal{M}z)\rangle + \langle \mathcal{D}y, (\omega^{-1})'\omega\mathcal{M}z\rangle \\ & +\langle(\omega^{-1})'\omega y, (\omega^{-1})\omega\mathcal{M}z\rangle]\, ds.\end{aligned}$$

Since Φ is a bounded bilinear functional in $W_2^1(\mathbb{R},\mathbb{R}^N)$, as in Lemma 5.2.11 there exists a bounded linear operator $\mathcal{K} : W_2^1(\mathbb{R},\mathbb{R}^N) \to W_2^1(\mathbb{R},\mathbb{R}^N)$ satisfying

$$\Phi(y,z) = \langle y, \mathcal{K}z\rangle_{W_2^1(\mathbb{R},\mathbb{R}^N)}. \tag{5.92}$$

Now $(\omega^{-1})'\omega = -\frac{1}{2}\mu^{-1}\mu'$ $(=: -\frac{1}{2}\nu$ in (5.7)), which is square integrable. Hence applying Lemma 5.2.10 as shown in the proof of Lemma 5.2.11 gives that $\mathcal{K}$ is compact in $W_2^1(\mathbb{R},\mathbb{R}^N)$. Part (ii) above shows that $\mathcal{K}_\natural$ (defined in (5.90)) is compact in $W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)$. Thus from (5.91) and (5.92),

$$\langle u, \tilde{\mathcal{M}}v\rangle_{W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)} = \langle u, \mathcal{M}_\natural v\rangle_{W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)} + \langle u, \mathcal{K}_\natural v\rangle_{W_{2,\mu}^1(\mathbb{R},\mathbb{R}^N)}.$$

So $\tilde{\mathcal{M}} = \mathcal{M}_\natural + \mathcal{K}_\natural$ and $K_\natural$ is compact, as required.

□

Lemma 5.2.13 *Let A and B be as in Lemma 5.2.11. Then there exists a bounded linear self-adjoint positive-definite operator $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, such that for $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,*

$$(L_\mu u)(S_\mu u) \geq \|u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta_\mu(u), \tag{5.93}$$

where $\theta_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to \mathbb{R}$ is such that $\theta_\mu(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and for $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, $L_\mu u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$ is defined by

$$(L_\mu u)(v) = \int_{\mathbb{R}} [\langle A(\mathcal{D}u)(s), (\mathcal{D}v)(s)\rangle - \langle B(s)u(s), v(s)\rangle]\mu(s)\, ds \tag{5.94}$$

for $v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.

Proof. Define the operator $T_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ by

$$(T_\mu u)(s) = \omega^{-1}(s)(T(\omega u))(s), \quad s \in \mathbb{R}, \; u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N), \tag{5.95}$$

where ω is defined in (5.7) and $T : W^1_2(\mathbb{R}, \mathbb{R}^N) \to W^1_2(\mathbb{R}, \mathbb{R}^N)$ is as defined in (5.76) in the proof of Lemma 5.2.11.
Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and write $w = \omega u$. Then $w \in W^1_2(\mathbb{R}, \mathbb{R}^N)$ and

$$\begin{aligned}(L_\mu u)(T_\mu u) &= \int_{\mathbb{R}} [\langle A\mathcal{D}w, \mathcal{D}(Tw)\rangle - \langle Bw, Tw\rangle]\, ds \\ &\quad + \int_{\mathbb{R}} [\langle A(\omega^{-1})'\omega w, \mathcal{D}(Tw)\rangle + \langle A\mathcal{D}w, (\omega^{-1})'\omega Tw\rangle \\ &\qquad \langle A(\omega^{-1})'\omega w, (\omega^{-1})'\omega Tw\rangle]\, ds. \end{aligned} \tag{5.96}$$

Estimate (5.80) gives that there exists θ such that $\theta(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W^1_2(\mathbb{R}, \mathbb{R}^N)$ as $n \to \infty$, and

$$\begin{aligned}\int_{\mathbb{R}} [\langle A\mathcal{D}w, \mathcal{D}(Tw)\rangle - \langle Bw, Tw\rangle]\, ds &\geq \frac{1}{2}\|w\|^2_{W^1_2(\mathbb{R},\mathbb{R}^N)} + \theta(w) \\ &\geq \eta\|u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \tilde{\theta}_\mu(u)\end{aligned}$$

where $(2\eta)^{-\frac{1}{2}}$ is the norm of the multiplication operator $\omega^{-1} : W^1_2(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and $\tilde{\theta}_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to \mathbb{R}$ satisfies $\tilde{\theta}_\mu(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.

From the observation that $(\omega^{-1})'\omega$ is square integrable, it follows using Lemma 5.2.10 (as in the proof of Lemma 5.2.12) that the second integral on the right-hand side of (5.96) tends to zero as $u_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Hence there is a functional θ_μ on $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ with $\theta_\mu(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, such that for $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,

$$(L_\mu u)(T_\mu u) \geq \eta \|u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta_\mu(u). \tag{5.97}$$

To complete the proof, it remains to show that there exist bounded linear operators $S_\mu, K_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ such that S_μ is self-adjoint and positive-definite, K_μ is compact, and

$$T_\mu = \eta S_\mu + K_\mu. \tag{5.98}$$

Constructing operators in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ from operators in $W^1_2(\mathbb{R}, \mathbb{R}^N)$ using (5.90), we find from (5.81) that

$$T_\natural = \frac{1}{2} S_\natural + K_\natural. \tag{5.99}$$

By Lemma 5.2.12, $S_\natural$ is a bounded linear self-adjoint positive-definite operator in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $K_\natural$ is compact in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Definition (5.95) and part (iv) of Lemma 5.2.12 give that $T_\natural - T_\mu$ is compact in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Hence S_μ, K_μ : $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ given by $S_\mu = (2\eta)^{-1} S_\natural$ and $K_\mu = K_\natural + (T_\mu - T_\natural)$ satisfy (5.98). The result follows.

□

Conditions **(f1)** and **(f2)** imply that $df[S]$, $df[T] \in P^{N\times N}$. With condition **(f3)**, this yields that $df[S]$ and $df[T]$ fulfil the conditions on the matrix B_0 in Lemma 5.2.9. Let $B_1 = df[S]$, $B_2 = df[T]$, and define the matrix B by (5.73). Then let $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ be the bounded linear self-adjoint positive-definite operator whose existence is guaranteed by Lemma 5.2.13 corresponding to this matrix B and the positive-definite diagonal matrix A as in the definition of the operator P_R given in (5.32). It will be shown that the operator S_μ satisfies the requirements of Theorem 5.2.7.

Proof of Theorem 5.2.7. Let the sequence $\{u_n\}_{n=1}^\infty \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ be such that $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ as $n \to \infty$, and set $v_n = u_n - u_0$. Then for each

$\tau \in [0,1]$, (5.32) gives that

$$\begin{aligned}(P_R(\tau,u_n))(S_\mu v_n) &= \int_{\mathbb{R}} [\langle A(\mathcal{D}u_n), \mathcal{D}(S_\mu v_n)\rangle + \langle A(\mathcal{D}u_n), \nu(S_\mu v_n)\rangle \\ &\quad -\langle A\psi'' + c(u_n)[\mathcal{D}u_n + \psi'] + \tau\sigma_R G(u_n+\psi, \mathcal{D}u_n+\psi')[\mathcal{D}u_n+\psi'] \\ &\quad +f(u_n+\psi), S_\mu v_n\rangle]\mu\, ds \qquad (5.100)\end{aligned}$$

where ν is as defined in (5.7). Consider the last part of the right-hand side of (5.100). We will prove that

$$\int_{\mathbb{R}} \langle f(u_n(s)+\psi(s)), (S_\mu v_n)(s)\rangle \mu(s)\, ds = \int_{\mathbb{R}} \langle B(s)v_n(s), (S_\mu v_n)(s)\rangle \mu(s)\, ds + \theta(v_n), \tag{5.101}$$

where $\theta(v_n) \to 0$ as $n \to \infty$. With this aim, note that

$$f(u_n(s)+\psi(s)) = f(u_0(s)+\psi(s)) + B_n(s)v_n(s), \tag{5.102}$$

where

$$B_n(s) := \int_0^1 df[tv_n(s)+u_0(s)+\psi(s)]\, dt, \tag{5.103}$$

so

$$\begin{aligned}&\int_{\mathbb{R}} \langle f(u_n+\psi), S_\mu v_n\rangle \mu\, ds - \int_{\mathbb{R}} \langle Bv_n, S_\mu v_n\rangle \mu\, ds \\ &= \int_{\mathbb{R}} \langle f(u_0+\psi), S_\mu v_n\rangle \mu\, ds + \int_{\mathbb{R}} \langle (B_n - df[u_0+\psi])v_n, S_\mu v_n\rangle \mu\, ds \\ &\qquad \int_{\mathbb{R}} \langle (df[u_0+\psi] - B)v_n, S_\mu v_n\rangle \mu\, ds. \qquad (5.104)\end{aligned}$$

Since $v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, $\|v_n\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}$ is bounded independently of n, and by the Sobolev Embedding Theorem, $v_n(s) \to 0$ uniformly for s in each compact subset of $\mathbb{R}$. In fact, since $\mu(s) \to \infty$ as $|s| \to \infty$ and estimate (5.10) holds, $v_n(s) \to 0$ uniformly on $\mathbb{R}$. Hence

$$B_n(s) = \int_0^1 df[tv_n(s)+u_0(s)+\psi(s)]\, dt \to \int_0^1 df[u_0(s)+\psi(s)]\, dt = df[u_0(s)+\psi(s)]$$

uniformly on $\mathbb{R}$. So since $\{\|v_n\|_{W^1_2(\mathbb{R},\mathbb{R}^N)}\}$ is bounded, the second term on the right of (5.104) tends to zero as $n \to \infty$.

Also, estimate (5.10) implies that $u_0(s)$ is uniformly bounded on $\mathbb{R}$ and $u_0(s) \to 0$ as $|s| \to \infty$, so because $f \in C^1(\mathbb{R}, \mathbb{R}^N)$, $df[u_0(s)+\psi(s)] \to df[S]$ as $s \to -\infty$ and $df[u_0(s)+\psi(s)] \to df[T]$ as $s \to \infty$. Hence $df[u_0(s)+\psi(s)] - B(s) \to 0$ as $|s| \to \infty$. Let $\epsilon > 0$. Then since $\{\|v_n\|_{L_2(\mathbb{R},\mathbb{R}^N)}\}$ is bounded, there exists some $\kappa > 0$ such that

$$\left| \int_{|s|>\kappa} \langle (df[u_0+\psi] - B)v_n, S_\mu v_n \rangle \mu \, ds \right| < \frac{\epsilon}{2}$$

for every $n \in \mathbb{N}$. Moreover, $v_n \to 0$ uniformly on $[-\kappa, \kappa]$ and $df[u_0(s)+\psi(s)] - B(s)$ is uniformly bounded, so there exists $N_0 \in \mathbb{N}$ so that $n \geq N_0$ implies that

$$\left| \int_{|s|\leq\kappa} \langle (df[u_0+\psi] - B)v_n, S_\mu v_n \rangle \mu \, ds \right| < \frac{\epsilon}{2}.$$

Hence the third term on the right-hand side of (5.104) also tends to zero.

To treat the first term on the right-hand side of (5.104), recall from the proof of Lemma 5.2.4 that $f(u_0+\psi) \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ whenever $u_0 \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Hence tha mapping $v \mapsto \int_{\mathbb{R}} \langle f(u_0+\psi), S_\mu v \rangle \mu \, ds$ is a member of $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$, and so the first term on the right of (5.104) tends to zero as $n \to \infty$. Thus (5.102) holds. Return to (5.100). The first term on the right-hand side is

$$\int_{\mathbb{R}} \langle A(\mathcal{D}u_n), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds = \int_{\mathbb{R}} \langle A(\mathcal{D}v_n), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds + \int_{\mathbb{R}} \langle A(\mathcal{D}u_0), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds.$$

$$\int_{\mathbb{R}} \langle A(\mathcal{D}u_n), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds = \int_{\mathbb{R}} \langle A(\mathcal{D}v_n), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds + \int_{\mathbb{R}} \langle A(\mathcal{D}u_0), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds \qquad (5.105)$$

Since the mapping $v \mapsto \int_{\mathbb{R}} \langle A(\mathcal{D}u_0), \mathcal{D}(S_\mu v) \rangle \mu \, ds$

is in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$, the second integral on the right of (5.105) tends to zero as $v_n \rightharpoonup 0$. The Cauchy-Schwarz inequality now gives that the second term on the right-hand side of (5.100) satisfies

$$\left| \int_{\mathbb{R}} \langle A(\mathcal{D}u_n), \nu S_\mu v_n \rangle \mu \, ds \right| \leq \left(\int_{\mathbb{R}} \|A\mathcal{D}u_n\|^2 \mu \, ds \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|y_n\|^2 \nu^2 \, ds \right)^{\frac{1}{2}} \qquad (5.106)$$

where $y_n = \omega S_\mu v_n$ (where ω is as defined in (5.7)).

The first factor on the right-hand side of (5.106) is bounded independently of n, since $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Since $S_\mu v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, the fact that multiplication by ω is a bounded linear operator from $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ to $W^1_2(\mathbb{R},\mathbb{R}^N)$ yields that $y_n \rightharpoonup 0$ in $W^1_2(\mathbb{R},\mathbb{R}^N)$, and hence $\{y_n(s)\}$ is uniformly bounded on $\mathbb{R}$. Furthermore, $y_n(s) \to 0$ uniformly on every compact subset of $\mathbb{R}$. Thus since ν^2 is integrable, Lebesgue's Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}} \|y_n\|^2 \nu^2 \, ds \to 0$$

as $n \to \infty$, and hence the left-hand side of (5.106) tends to zero.

Clearly, $\int_{\mathbb{R}} \langle A\psi'', S_\mu v_n\rangle \mu \, ds \to 0$ as $v_n \rightharpoonup 0$. Since $\{u_n\}$ is bounded in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, Lemma 5.2.3 yields that $c(u_n)$ is bounded independently of n, so that

$$c(u_n) \int_{\mathbb{R}} \langle \psi', S_\mu v_n\rangle \mu \, ds \to 0$$

as $n \to \infty$.

Now hypothesis **(G2)** and the facts that $|\tau\sigma_R(s)| \leq 1$ for each s and σ_R is supported in $[-R-1, R+1]$ give that

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \tau\sigma_R \langle G(u_n+\psi, \mathcal{D}u_n+\psi')[\mathcal{D}u_n+\psi'], S_\mu v_n\rangle \mu \, ds \right| \qquad (5.107)\\
&\leq \int_{|s|\leq R+1} |\langle G(u_n+\psi, \mathcal{D}u_n+\psi')(\mathcal{D}u_n+\psi'), S_\mu v_n\rangle| \mu \, ds \\
&\leq \int_{|s|\leq R+1} \|G(u_n+\psi, \mathcal{D}u_n+\psi')(\mathcal{D}u_n+\psi')\| \|S_\mu v_n\| \mu \, ds \\
&\leq \int_{|s|\leq R+1} \{\beta(u_n+\psi) + \gamma(u_n+\psi)\|\mathcal{D}u_n+\psi'\|\} \|S_\mu v_n\| \mu \, ds \\
&\leq \beta_0 \int_{|s|\leq R+1} \|S_\mu v_n\| \mu \, ds + \gamma_0 \int_{|s|\leq R+1} \|\mathcal{D}u_n+\psi'\| \|S_\mu v_n\| \mu \, ds \\
&\leq \left\{ \beta_0 \sqrt{2(R+1)} + \gamma_0 \left(\int_{|s|\leq R+1} \|\mathcal{D}u_n+\psi'\|^2 \mu \, ds \right)^{\frac{1}{2}} \right\} \left(\int_{|s|\leq R+1} \|S_\mu v_n\|^2 \mu \, ds \right)^{\frac{1}{2}}. \qquad (5.108)
\end{aligned}$$

Here $\{u_n(s)\}$ is bounded independently of n and of s, so since the functions β and γ are continuous, there exist $\beta_0, \gamma_0 > 0$ such that $\beta(u_n(s)+\psi(s)) < \beta_0$ and $\gamma(u_n(s)+\psi(s)) < \gamma_0$ for all $n \in \mathbb{N}$, $s \in \mathbb{R}$.

Since $\{u_n\}$ is bounded in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ and $\psi \in C^\infty(\mathbb{R},\mathbb{R}^N)$, the first factor in (5.108) is bounded independently of n. Also, $v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ implies that $S_\mu v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, so $(S_\mu v_n)(s) \to 0$ uniformly for $|s| < R+1$. Hence the second factor in (5.108) tends to zero, and thus (5.107) tends to zero as $n \to \infty$ *uniformly* for $\tau \in [0,1]$.

It remains to show that

$$\int_{\mathbb{R}} \langle \mathcal{D}u_n, S_\mu v_n \rangle \mu \, ds \to 0 \quad \text{as } n \to \infty. \tag{5.109}$$

Use of (5.98) gives that

$$\begin{aligned}\int_{\mathbb{R}} \langle \mathcal{D}u_n, S_\mu v_n \rangle \mu \, ds &= \int_{\mathbb{R}} \langle \mathcal{D}v_n, \eta^{-1} T_\mu v_n \rangle \mu \, ds - \int_{\mathbb{R}} \langle \mathcal{D}v_n, \eta^{-1} K_\mu v_n \rangle \mu \, ds \\ &\quad + \int_{\mathbb{R}} \langle \mathcal{D}u_0, \eta^{-1}(T_\mu - K_\mu) v_n \rangle \mu \, ds. \end{aligned} \tag{5.110}$$

The second term on the right of (5.110) tends to zero as $v_n \rightharpoonup 0$ since $\mathcal{D}v_n$ is bounded in $L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and since K_μ is compact, $K_\mu v_n \to 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. The third term is clearly a bounded linear functional of v_n, so tends to zero as $v_n \rightharpoonup 0$.

Now let $w_n = \omega v_n$ and recall the definition of T_μ in (5.95). Then with T as defined in (5.76),

$$\int_{\mathbb{R}} \langle \mathcal{D}v_n, T_\mu v_n \rangle \mu \, ds = \int_{\mathbb{R}} \langle \mathcal{D}w_n, T w_n \rangle \, ds - \int_{\mathbb{R}} \langle \omega' v_n, T w_n \rangle \, ds. \tag{5.111}$$

Since $\omega' v_n = \frac{1}{2} \nu w_n$,

$$\begin{aligned}\left| \int_{\mathbb{R}} \langle \omega' v_n, T w_n \rangle \, ds \right| &= \frac{1}{2} \left| \int_{\mathbb{R}} \langle T w_n, \nu w_n \rangle \, ds \right| \\ &\le \left(\int_{\mathbb{R}} \nu^2 \|w_n\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|T w_n\|^2 \, ds \right)^{\frac{1}{2}}. \end{aligned} \tag{5.112}$$

Lebesgue's Dominated Convergence Theorem yields that the first factor in (5.112) tends to zero as $n \to \infty$, since ν^2 is integrable, $\{w_n(s)\}$ is uniformly bounded independently of n, and $w_n(s) \to 0$ uniformly on each compact subset of $\mathbb{R}$. The second factor is bounded independently of n since w_n is bounded in $L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and hence the second term on the right of (5.111) tends to zero.
By virtue of the definition of T in (5.76),

$$\int_{\mathbb{R}} \langle \mathcal{D}w_n, T w_n \rangle \, ds = \sum_{i=1}^{2} \int_{\mathbb{R}} [\langle \mathcal{D}(\phi_i w_n), T_i \phi_i w_n \rangle - \langle \phi_i' w_n, T_i \phi_i w_n \rangle] \, ds \tag{5.113}$$

where ϕ_i, T_i are defined in the statement and proof of Lemma 5.2.11 respectively. Since $\phi_i(s) = 0$ when $|s| > 1$ for each i,

$$\int_{\mathbb{R}} \langle \phi_i w_n, T_i \phi_i w_n \rangle \, ds \to 0 \quad \text{as } n \to \infty,$$

since $T_i(\phi_i w_n) \to 0$ uniformly for $|s| \leq 1$.

Now for each i, n, $\phi_i w_n \in W_2^1(\mathbb{R}, \mathbb{R}^N)$. Let Ψ_i, C_i correspond to T_i as Ψ, C correspond to T_0 in the proof of Lemma 5.2.9. Then by Parseval's formula,

$$\begin{aligned}
\int_{\mathbb{R}} \langle \mathcal{D}(\phi_i w_n), T_i(\phi_i w_n) \rangle \, ds &= \mathrm{Re} \int_{\mathbb{R}} \langle \mathcal{D}(\phi_i w_n), \Psi_i(\phi_i w_n) \rangle \, ds \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle \widehat{\mathcal{D}(\phi_i w_n)}, \widehat{\Psi_i(\phi_i w_n)} \rangle_{\mathbb{C}^N} \, d\xi \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle i\xi \widehat{\phi_i w_n}, R_i(\xi) \widehat{\phi_i w_n} \rangle_{\mathbb{C}^N} \, d\xi \\
&= 2\mathrm{Re} \int_{\mathbb{R}} \left\{ \int_0^\infty \langle i\xi \widehat{\phi_i w_n}, e^{tC_i^*(\xi)} e^{tC_i(\xi)} \widehat{\phi_i w_n} \rangle_{\mathbb{C}^N} \, dt \right\} d\xi \\
&= 2\mathrm{Re} \int_{\mathbb{R}} \left\{ \int_0^\infty \langle i\xi e^{tC_i(\xi)} \widehat{\phi_i w_n}, e^{tC_i(\xi)} \widehat{\phi_i w_n} \rangle_{\mathbb{C}^N} \, dt \right\} d\xi \\
&= 2\mathrm{Re} \int_0^\infty \left\{ \int_{\mathbb{R}} \langle i\xi e^{tC_i(\xi)} \widehat{\phi_i w_n}, e^{tC_i(\xi)} \widehat{\phi_i w_n} \rangle_{\mathbb{C}^N} \, d\xi \right\} dt.
\end{aligned} \tag{5.114}$$

The last equality follows by Fubini's Theorem.
Now for each $t \in (0, \infty)$, define $H(t) : W_2^1(\mathbb{R}, \mathbb{R}^N) \to \tilde{W}_2^1(\mathbb{R}, \mathbb{C}^N)$ by

$$\widehat{H(t)u}(\xi) = e^{tC_i(\xi)} \hat{u}(\xi), \quad \xi \in \mathbb{R}, u \in W_2^1(\mathbb{R}, \mathbb{R}^N), \tag{5.115}$$

which is well-defined since $e^{tC_i(\xi)}, t \in (0, \infty)$ is bounded independently of ξ (by (5.66). Then for fixed t,

$$\begin{aligned}
&\mathrm{Re} \int_{\mathbb{R}} \langle i\xi e^{tC_i(\xi)} \widehat{\phi_i w_n}, e^{tC_i(\xi)} \widehat{\phi_i w_n} \rangle_{\mathbb{C}^N} \, d\xi \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle i\xi \widehat{H(t)(\phi_i w_n)}, \widehat{H(t)(\phi_i w_n)} \rangle_{\mathbb{C}^N} \, d\xi \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle (\widehat{\mathcal{D}(H(t)(\phi_i w_n))}, \widehat{H(t)(\phi_i w_n)} \rangle_{\mathbb{C}^N} \, d\xi \\
&= \mathrm{Re} \int_{\mathbb{R}} \langle \mathcal{D}(H(t)(\phi_i w_n)), H(t)(\phi_i w_n) \rangle_{\mathbb{C}^N} \, ds
\end{aligned} \tag{5.116}$$

The expression in (5.116) equals zero because $H(s)(\phi_i w_n) \in \tilde{W}_2^1(\mathbb{R}, \mathbb{C}^N)$. (It follows using integration by parts that for $\rho \in \tilde{C}_0^\infty(\mathbb{R}, \mathbb{C}^N)$,

$$\mathrm{Re} \int_{\mathbb{R}} \langle \rho', \rho \rangle_{\mathbb{C}^N} \, ds = 0,$$

and thus by the density of $\tilde{C}_0^\infty(\mathbb{R}, \mathbb{C}^N)$ in $\tilde{W}_2^1(\mathbb{R}, \mathbb{C}^N)$,

$$\mathrm{Re} \int_{\mathbb{R}} \langle \mathcal{D}u, u \rangle_{\mathbb{C}^N} \, ds = 0$$

for $u \in \tilde{W}_2^1(\mathbb{R}, \mathbb{C}^N)$.) Hence the first term on the right-hand side of (5.113) equals zero, and so

$$\int_{\mathbb{R}} \langle \mathcal{D}u_n, S_\mu v_n \rangle \mu \, ds \to 0.$$

There thus exists a functional θ such that $\theta(u_n, u_0, \tau) \to 0$ as $u_n \rightharpoonup u_0$ in $W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)$ uniformly for $\tau \in [0,1]$, and by (5.101) and (5.105),

$$\begin{aligned}
(P_R(\tau, u_n))(S_\mu v_n) &= \int_{\mathbb{R}} [\langle A(\mathcal{D}v_n)(s), \mathcal{D}(S_\mu v_n)(s) \rangle - \langle B(s) v_n(s), (S_\mu v_n(s)) \rangle] \mu(s) \, ds \\
&\qquad + \theta(u_n, u_0, \tau) \\
&= (L_\mu v_n)(S_\mu v_n) + \theta_\mu(v_n) + \theta(u_n, u_0, \tau) \\
&\geq \|u_n - u_0\|^2_{W_{2,\mu}^1(\mathbb{R}, \mathbb{R}^N)} + \theta_\mu(v_n) + \theta(u_n, u_0, \tau).
\end{aligned}$$

Here L_μ, S_μ and θ_μ are as in Lemma 5.2.11. The result follows.

□

Chapter 6

Uniform *a priori* estimates for monotone solutions of the approximate system

For $R > 0$, define

$$\mathcal{M}_R := \left\{ u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N) : \begin{array}{c} u \text{ satisfies (5.31) for this } R \\ \text{for some } \tau \in [0,1], \text{and} \\ w = u + \psi \text{ is monotone} \end{array} \right\}. \tag{6.1}$$

The goal of this chapter is to prove that $\mathcal{M}_R$ is bounded in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ independently of $R > 0$. In doing so, we will prove some *a priori* estimates on solutions of the general travelling-wave system (2.7), which are of independent interest. The fact that $\mathcal{M}_R$ is bounded *for a given* R will be used in Chapter 7 to construct a bounded set $\Omega \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, depending on R, that satisfies (5.58), and is such that there are no solutions u of (5.31) with $w = u + \psi$ non-monotone in $\overline{\Omega}$. Existence of a monotone solution to (5.30) will then be established for each $R > 0$. Chapter 7 concludes with the proof of the existence of a monotone solution to (1.8) via a limiting procedure, which stems from the fact that the bound on $\mathcal{M}_R$ is independent of $R > 0$.

6.1 *A priori* estimates for a general system

We will first consider solutions $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ of

$$A(u'' + \psi'') + c[u' + \psi'] + \sigma G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0, \quad (6.2)$$

where f satisfies **(f1)** - **(f4)**, G satisfies **(G1)-(G3)**, and $\sigma \in C^1(\mathbb{R}, [0,1])$. The velocity $c \in \mathbb{R}$ is for now regarded as a *parameter*, and not the functional (5.25). The representation of c as a functional will be reintroduced in the final stage of obtaining the uniform *a priori* bound for monotone solutions. This procedure will also rely on the fact that the set of velocities for which there is a monotone solution to (6.2) is uniformly bounded.

We begin by obtaining an *a priori* estimate for the speed of monotone solutions. The initial step, which will also be useful elsewhere, is to obtain estimates for the derivatives of solutions of (6.2) that are independent of c and the choice of $\sigma \in C^1(\mathbb{R}, [0,1])$. Note that it is *not* assumed yet that the solutions are monotone. We first consider the behaviour of derivatives of any solution $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ of (6.2).

Lemma 6.1.1 *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (6.2) for some $c \in \mathbb{R}$ and $\sigma \in C^1(\mathbb{R}, [0,1])$. Then*

$$u, u', u'' \in L_\infty(\mathbb{R}, \mathbb{R}^N),$$

and there exists $K > 0$, depending on u, f, G and c, such that

$$\|u'(s)\| \leq \frac{K}{\sqrt{\mu(s)}} \quad \textit{for each} \quad s \in \mathbb{R}, \quad (6.3)$$

and hence $\|u'(s)\| \to 0$ as $n \to \infty$.

Proof. That $u \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ follows from estimate (5.10). Since u satisfies (6.2),

$$u'' = -\psi'' - A^{-1}\{c[u' + \psi'] + \sigma G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi)\}. \quad (6.4)$$

Clearly $-\psi'' - cA^{-1}[u' + \psi'] \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, because $u' \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and ψ' and ψ'' have compact support. Also, estimate (5.10) together with Lemma 5.0.3 yield that $A^{-1}\{\sigma G(u+\psi, u'+\psi')(u'+\psi')\} \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and it was shown in the proof of Lemma 5.2.4 that $f(u+\psi) \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Hence $u'' \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and $u' \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. That $u' \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ and estimate (6.3) thus follow from (5.10). That $u'' \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ follows from (6.2), the boundedness of ψ, ψ', u and u' and the continuity of f and G.

□

Theorem 6.1.2 *Let $M > 0$ and $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (6.2) for some $c \in \mathbb{R}$ and $\sigma \in C^1(\mathbb{R}, [0,1])$ and such that $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < M$, where $w = u + \psi$. Then there exist $N_1, N_2 > 0$, dependent upon M but independent of w, c and σ, such that*

$$\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < N_1 \text{ and } \|w''\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < N_2.$$

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(s) := \langle Aw'(s), w'(s) \rangle, s \in \mathbb{R}$. Then h is continuously differentiable, and $h \geq 0$ for all s, since A is positive-definite. Also, Lemma 6.1.1 implies that $h(s) \to 0$ as $|s| \to \infty$, since $\psi'(s) = 0$ when $|s| > 1$. So h attains a maximum at a point $s_0 \in \mathbb{R}$ where $h'(s_0) = 0$. Since A is symmetric,

$$\langle Aw''(s_0), w'(s_0) \rangle = 0,$$

so taking the inner product of the left-hand side of (6.2) evaluated at s_0 with $w'(s_0)$ yields that

$$c\|w'(s_0)\|^2 + \sigma(s_0)\langle G(w(s_0), w'(s_0))w'(s_0), w'(s_0)\rangle + \langle f(w(s_0)), w'(s_0)\rangle = 0. \quad (6.5)$$

Now by hypothesis **(G2)**,

$$|\sigma(s_0)\langle G(w(s_0), w'(s_0))w'(s_0), w'(s_0)\rangle| \leq (\beta(w(s_0)) + \gamma(w(s_0))\|w'(s_0)\|)\|w'(s_0)\|,$$

so

$$\begin{aligned} |c|\|w'(s_0)\|^2 &= |\sigma(s_0)\langle G(w(s_0), w'(s_0))w'(s_0), w'(s_0)\rangle + \langle f(w(s_0)), w'(s_0)\rangle| \\ &\leq (\beta(w(s_0)) + \gamma(w(s_0))\|w'(s_0)\|)\|w'(s_0)\| + \|f(w(s_0))\|\|w'(s_0)\|. \quad (6.6) \end{aligned}$$

Since $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < M$ and β, γ and f are continuous, there exist $\beta_0, \gamma_0, \alpha_0 > 0$, depending *only* upon M, such that

$$\beta(w(s)) < \beta_0, \quad \gamma(w(s)) < \gamma_0 \ \text{ and } \ \|f(w(s))\| < \alpha_0 \tag{6.7}$$

for all $s \in \mathbb{R}$. Hence (6.6) gives that if $\|w'(s_0)\| > 0$,

$$(|c| - \gamma_0)\|w'(s_0)\| \leq \beta_0 + \alpha_0. \tag{6.8}$$

Consider first the possibility that $|c| \geq 2\gamma_0$. Then

$$\|w'(s_0)\| \leq \frac{1}{\gamma_0}(\beta_0 + \alpha_0), \tag{6.9}$$

which is trivial if $w'(s_0) = 0$, and otherwise is a consequence of (6.8). So, as h attains a maximum at s_0,

$$h(s) \leq h(s_0) = \langle Aw'(s_0), w'(s_0)\rangle \leq a_1(\frac{1}{\gamma_0}(\beta_0 + \alpha_0))^2 \ \text{ for all } \ s \in \mathbb{R}, \tag{6.10}$$

where $a_1 := \max\limits_{1\leq i\leq N}\{A_1, \ldots, A_N\}$. Let $a_2 := \min\limits_{1\leq i\leq N}\{A_1, \ldots, A_N\}$. Then $a_2 > 0$ and $h(s) \geq a_2\|w'(s)\|^2$ for each $s \in \mathbb{R}$, so (6.10) yields

$$|c| \geq 2\gamma_0 \Rightarrow \|w'(s)\| \leq \left(\frac{a_1}{a_2}\right)^{\frac{1}{2}} \frac{1}{\gamma_0}(\beta_0 + \alpha_0) \ \text{ for all } \ s \in \mathbb{R}. \tag{6.11}$$

Suppose now that $|c| \leq 2\gamma_0$. Lemma 6.1.1 implies that w' and w'' are uniformly bounded on $\mathbb{R}$. So Landau's inequality and the fact that $u = w - \psi$ satisfies (6.2) together yield that

$$\begin{aligned}
\|w'\|^2_{L_\infty(\mathbb{R},\mathbb{R}^N)} &\leq 4\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\|A^{-1}(cw' + \sigma G(w,w')w' + f(w))\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \\
&\leq \frac{4M}{a_2}(2\gamma_0\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} + \beta_0 + \gamma_0\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} + \alpha_0)
\end{aligned}$$

since $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq M, |c| \leq 2\gamma_0$, and **(G2)** and (6.7) hold.
Hence

$$|c| \leq 2\gamma_0 \Rightarrow \|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq \max\{1, \frac{4M}{a_2}(3\gamma_0 + \beta_0 + \alpha_0)\}. \tag{6.12}$$

The existence of $N_1 > 0$ as in the statement of the theorem follows from (6.11) and (6.12).

To obtain the corresponding bound for w'', observe that the i^{th} equation of (6.2) yields that for each $s \in \mathbb{R}$,

$$|w_i''(s)| \leq A_i^{-1} \left\{ \sup_{s \in \mathbb{R}} |cw_i'(s)| + \sup_{s \in \mathbb{R}} |G_i(w(s), w'(s))w_i'(s)| + \sup_{s \in \mathbb{R}} |f(w(s))| \right\}. \tag{6.13}$$

Since $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq M$, $\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq N_1$, and f and G are continuous, the second and third terms on the right-hand side of (6.13) are bounded by some $\alpha > 0$ dependent only on M.

If $c = 0$, the existence of $N_2 > 0$ as required follows immediately. Otherwise, note that $cw_i'(s) \to 0$ as $|s| \to \infty$, so as in the first part of this proof, $|cw_i'|$ attains a maximum at some $s_0 \in \mathbb{R}$ where $(cw_i')'(s_0) = cw_i''(s_0) = 0$. Since $c \neq 0$, $w_i''(s_0) = 0$, so from the i^{th} equation of (6.2),

$$cw_i'(s_0) + \sigma(s_0)G_i(w(s), w'(s))w_i'(s) + f_i(w(s_0)) = 0. \tag{6.14}$$

As argued in the last paragraph, the second and third terms of (6.14) are bounded by $\alpha > 0$, so $cw_i'(s_0)$ is bounded by $\alpha > 0$, dependent only on M. Since $|cw_i'|$ attains its maximum at s_0, the existence of $N_2 > 0$ as in the statement of the theorem follows from (6.13).

□

To obtain an *a priori* estimate for the speed of monotone solutions to (6.2), some preliminary results on the signs of the components of the function f are also required. For clarity, we specify conditions on f in each of the following two lemmas.

Lemma 6.1.3 *Let $f \in C^1(\mathbb{R}, \mathbb{R}^N)$ satisfy* **(f1)**, **(f2)** *and* **(f3)**. *Let $p, q > 0$ denote Perron-Frobenius eigenvectors of $df[S]$ and $df[T]$ respectively. Then there exist $t, \zeta > 0$ such that for each $i \in \{1, \ldots, N\}$,*

$$x \in \Gamma_i := \{y \in \mathbb{R}^N : S \leq y \leq S + tp, y_i = S_i + tp_i\} \Rightarrow f_i(x) < -\zeta < 0, \tag{6.15}$$

$$x \in \Lambda_i := \{y \in \mathbb{R}^N : T - tq \leq y \leq T, y_i = T_i - tq_i\} \Rightarrow f_i(x) > \zeta > 0. \tag{6.16}$$

Proof. We prove (6.15); the argument for (6.16) is similar. Suppose, for contradiction, that for some $i \in \{1, \dots, N\}$, there is a sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}^+, t_n \downarrow 0$ as $n \to \infty$ such that $f_i(x^n) \geq 0$ for $x^n \in \mathbb{R}^N$ with $x_i^n = S_i + t_n p_i$ and $S_j \leq x_j^n \leq S_j + t_n p_j$ for $j \in \{1, \dots, N\}, i \neq j$. Clearly $x^n \to S$ as $n \to \infty$. Since f_i is differentiable at S and $f_i(S) = 0$,

$$f_i(x^n) = B_i(x^n - S) + R_i(x^n - S),$$

where $B_i := df_i[S]$ and $R_i : \mathbb{R} \to \mathbb{R}$ is such that $\dfrac{\|R_i(x^n - S)|}{\|x^n - S\|} \to 0$ as $n \to \infty$. Hence

$$B_i\left(\frac{x^n - S}{t_n}\right) + \frac{R_i(x^n - S)}{t_n} \geq 0 \tag{6.17}$$

since $t_n > 0$ for each n. Now $\{y^n\} := \left\{\dfrac{x^n - S}{t_n}\right\}$, a bounded sequence in $\mathbb{R}^N$, has a convergent subsequence, say $y^k \to y$. Also,

$$\frac{R_i(x^n - S)}{t_n} = \frac{R_i(x^n - S)}{\|x^n - S\|} \cdot \frac{\|x^n - S\|}{t_n} \to 0 \text{ as } n \to \infty,$$

since $\|x^n - S\| \leq t_n \|p\|$. So by (6.17),

$$B_i y \geq 0. \tag{6.18}$$

But for each k, $y_i^k = p_i$ and $0 \leq y_j^k \leq p_j$ for $i \neq j$. Moreover, **(f3)** yields that $B_i p < 0$, and $\dfrac{\partial f_i}{\partial x_j} > 0$ for $i \neq j$ by **(f1)**. Together, these observations imply that $B_i y < 0$, which contradicts (6.18). Since Γ_i is compact and $f_i > 0$ on Γ_i, there exists $\zeta > 0$ such that $f_i \geq \zeta > 0$ on Γ_i. The result follows.

□

The continuity of f leads to the following straightforward strengthening of Lemma 6.1.3.

Lemma 6.1.4 *Let $f, \Gamma_i, \Lambda_i, t, p, q$ be as in Lemma 6.1.3. Then there exists $\epsilon > 0$ such that for each $i \in \{1, \dots, N\}$, $\epsilon < tp_i, \epsilon < tq_i$, and*

$$x \in \Gamma_i, y \in \mathbb{R}^N \text{ such that } \max_{1 \leq k \leq N} |y_k - x_k| \leq \epsilon \Rightarrow f_i(y) < 0, \tag{6.19}$$

$$x \in \Lambda_i, y \in \mathbb{R}^N \ \ such\ that \max_{1\le k\le N} |y_k - x_k| \le \epsilon \Rightarrow f_i(y) > 0. \tag{6.20}$$

Proof. We show (6.19). Let $i \in \{1,\ldots,N\}$ and $x \in \Gamma_i$. By Lemma 6.1.3, $f_i(x) < 0$, so since f_i is continuous, there exists $\epsilon_x > 0$ such that $y \in \mathbb{R}^N, \max_{1\le k\le N}|x_k - y_k| < \epsilon_x \Rightarrow f_i(y) < 0$. Now the family of sets $\{y : \max_{1\le k\le N}|y_k - x_k| < \frac{\epsilon_x}{2}\}, x \in \Gamma_i$ is an open cover for Γ_i, so has a finite subcover $\{y : \max_{1\le k\le N}|y_k - x_k| < \frac{\epsilon_x}{2}\}, x \in \{x^1,\ldots,x^m\} \subset \Gamma_i, m \in \mathbb{N}$, since Γ_i is compact.

Let $\epsilon_i = \min_{1\le j\le m}\{\frac{\epsilon_{x^j}}{2}\}$. Now for $x \in \Gamma_i$, let $j \in \{1,\ldots,m\}$ be such that $\max_{1\le k\le N}|x_k - x_k^j| < \frac{\epsilon_{x^j}}{2}$. Then for y with $\max_{1\le k\le N}|y_k - x_k| < \epsilon_i$, $\max_{1\le k\le N}|y_k - x_k^j| < \epsilon_{x^j}$, so that $f_i(y) < 0$. Taking $\epsilon = \min_{1\le j\le N}\epsilon_j$ yields (6.20).

□

Lemmas 6.1.1 - 6.1.4 are now employed to bound the velocity of monotone solutions to (6.2), independently of the choice of $\sigma \in C^1(\mathbb{R},[0,1])$.

Theorem 6.1.5 *There exists $\gamma > 0$, depending only on f, G and A, such that if $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N)$ satisfies (6.2) for some $c \in \mathbb{R}$ and some $\sigma \in C^1(\mathbb{R},[0,1])$ and*

$$u'(s) + \psi'(s) > 0 \ \ for\ all\ \ s \in \mathbb{R},$$

then

$$|c| < \gamma. \tag{6.21}$$

Proof. Let u be as in the statement of the theorem. We will use the fact that $w(s) = u(s) + \psi(s)$ converges monotonically to S as $s \to -\infty$ to obtain a lower bound for c. The existence of an upper bound can be proved similarly, using the monotonic convergence of $w(s)$ to T as $s \to \infty$.

Since $w(s) \to S$ as $s \to -\infty$ and $u'(s) + \psi'(s) > 0$ for each s, there exist $s_0 \in \mathbb{R}$ and $i \in \{1,\ldots,N\}$ such that $w(s_0) > S$ and $w(s_0) \in \Gamma_i$, where Γ_i is defined in (6.15). Now $w(s_0) \in \Gamma_i \Rightarrow w_i(s_0) = tp_i$, and $tp_i > \epsilon > 0$, where ϵ is as in Lemma 6.1.4. Choose $s_1 \in \mathbb{R}$ such that $w_i(s_0) - w_i(s_1) = \epsilon$. Since w is increasing, $s_1 < s_0$. Integrating the i^{th} equation of (6.2) from s_1 to s_0 gives

$$A_i(w_i'(s_0) - w_i'(s_1)) + c\epsilon + \int_{s_1}^{s_0} \sigma(s)G_i(w(s), w'(s))w_i'(s)\,ds + \int_{s_1}^{s_0} f_i(w(s))\,ds = 0. \tag{6.22}$$

Since w is a *monotone* solution between S and T, there exists $M > 0$, independent of w, c and $\sigma \in C^1(\mathbb{R}, [0,1])$, such that

$$||w||_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq M. \tag{6.23}$$

Thus by Theorem 6.1.2, there exists $N_1 > 0$, independent of w, c and $\sigma \in C^1(\mathbb{R}, [0,1])$, such that

$$||w'||_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq N_1. \tag{6.24}$$

So

$$|A_i(w_i'(s_0) - w_i'(s_1))| \leq 2|A_i|N_1 \leq 2a_1N_1, \tag{6.25}$$

where $a_1 = \max\{A_1, \ldots, A_N\}$. Also, since G is continuous and (6.23) and (6.24) hold, there exists $\gamma_1 > 0$ such that for each $i \in \{1, \ldots, N\}$,

$$||G_i(w, w')||_{L_\infty(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})} \leq \gamma_1. \tag{6.26}$$

Hence

$$\begin{aligned} \left|\int_{s_1}^{s_0} \sigma(s)G_i(w(s), w'(s))w_i'(s)\,ds\right| &\leq \int_{s_1}^{s_0} |G_i(w(s), w'(s))||w_i'(s)|\,ds \\ &\leq \gamma_1 \int_{s_1}^{s_0} |w_i'(s)|\,ds \\ &= \gamma_1 \int_{s_1}^{s_0} w_i'(s)\,ds \\ &= \gamma_1\epsilon, \end{aligned} \tag{6.27}$$

since w is monotone. We now show that

$$f_i(w(s)) < 0 \text{ for } s_1 \leq s \leq s_0. \tag{6.28}$$

By Lemma 6.1.4, $f_i(y) < 0$ when $w_i(s_0) - \epsilon \leq y_i \leq w_i(s_0)$ and $y_k = w_k(s_0)$ for $k \in \{1, \ldots, N\}, k \neq i$. Thus by Lemma 4.0.11, $f_i(y) < 0$ when $w_i(s_0) - \epsilon \leq y_i \leq w_i(s_0)$ and $y_k \leq w_k(s_0), k \in \{1, \ldots, N\}, k \neq i$. That (6.28) holds then follows from the monotonicity of w. So from (6.22), (6.25), (6.27) and (6.28), we find that

$$c \geq -\left(\gamma_1 + \frac{2a_1N_1}{\epsilon}\right),$$

which gives a lower bound for the velocity c as required.

□

Next we will show that monotone solutions of (6.2) satisfy uniform exponential estimates in neighbourhoods of S and T, in the following sense. Note that these estimates are *independent* of the velocity c and of the choice of $\sigma \in C^1(\mathbb{R}, [0,1])$. The idea of the following proof is contained in [9].

Theorem 6.1.6 *There exist $\kappa, \alpha, \beta > 0, 0 < \delta < \frac{1}{2}\min_{1\leq i\leq N}\{T_i - S_i\}$, such that if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies (6.2) for some $c \in \mathbb{R}$ and some $\sigma \in C^1(\mathbb{R}, [0,1])$, and $u'(s) + \psi'(s) > 0$ for all $s \in \mathbb{R}$, then for $w = u + \psi$,*

$$\|w(s) - S\| \leq \kappa e^{\alpha(s-s_0)}, \quad \|w'(s)\| \leq \kappa e^{\alpha(s-s_0)}, \tag{6.29}$$

where $s_0 \in \mathbb{R}$ is the unique point such that $\|w(s_0) - S\| = \delta$ and $s \leq s_0$. Moreover,

$$\|w(s) - T\| \leq \kappa e^{-\beta(s-t_0)}, \quad \|w'(s)\| \leq \kappa e^{-\beta(s-t_0)}, \tag{6.30}$$

where $t_0 \in \mathbb{R}$ is the unique point such that $\|w(t_0) - T\| = \delta$ and $s \geq t_0$. In addition, for $w_0 \in \mathbb{R}^N$,

$$0 < \|w_0 - S\| \leq \delta \ \Rightarrow\ f(w_0) \neq 0, \tag{6.31}$$

$$0 < \|w_0 - T\| \leq \delta \ \Rightarrow\ f(w_0) \neq 0. \tag{6.32}$$

Proof. (6.30) and (6.32) will be proved; the argument for (6.29) and (6.31) is similar. Setting $v = w'$, the second order system of N equations (6.2) becomes the first order system of $2N$ equations,

$$\begin{pmatrix} w' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ -A^{-1}(cv + \sigma G(w,v)v + f(w)) \end{pmatrix}. \tag{6.33}$$

Now let

$$p(s) := \begin{pmatrix} w(s) \\ v(s) \end{pmatrix} \quad \text{and} \quad p_0 := \begin{pmatrix} T \\ 0 \end{pmatrix}. \tag{6.34}$$

By **(f2)** and **(G3)**, $f(T) = 0$ and $G(T, 0) = 0$. So since f and G are continuously differentiable, (6.33) can be rewritten as

$$p'(s) = B^c\{p(s) - p_0\} + H_\sigma(s, p(s) - p_0), \tag{6.35}$$

where $B^c \in M^{2N\times 2N}$ is given by

$$B^c := \begin{pmatrix} 0 & I \\ -A^{-1}df[T] & -cA^{-1} \end{pmatrix}, \tag{6.36}$$

and $H_\sigma : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is such that

(i) $\quad H_\sigma(s, 0) = 0$ for each $s \in \mathbb{R}$, and $\quad$ (6.37)

(ii) $\quad$ given $\epsilon > 0$, there exists $\eta > 0$, independent of $s \in \mathbb{R}$ and $\sigma \in C^1(\mathbb{R}, [0, 1])$, such that for $s \in \mathbb{R}$ and $p, q \in \mathbb{R}^{2N}$,

$$\|p - p_0\| < \eta \text{ and } \|q - p_0\| < \eta \Rightarrow \|H_\sigma(s, q - p_0) - H_\sigma(s, p - p_0)\| \le \epsilon\|p - q\|. \tag{6.38}$$

Note that u, w in the statement of the theorem satisfy the conditions of Theorem 6.1.5, so that

$$|c| \le \gamma, \tag{6.39}$$

where $\gamma > 0$ is independent of w and σ, as given by Theorem 6.1.5. Theorem 3.1.2 gives that B^c has N eigenvalues (counted according to algebraic multiplicity) in each *open* left-right half of the complex plane, for each $c \in \mathbb{R}$. Hence the bound (6.39), together with the fact that the eigenvalues of B^c for $|c| \le \gamma$ form a bounded set in $\mathbb{C}$ (Lemma 3.0.2), imply that these eigenvalues are bounded away from the imaginary axis independently of c satisfying (6.39). Hence there exist $\beta, \zeta > 0$ and bounded sets $\Omega_-, \Omega_+ \subset \mathbb{C}$ with simple piecewise continuous boundaries, such that if λ is a (complex) eigenvalue of B^c for some c satisfying (6.39), then

$$\operatorname{Re}\lambda < 0 \;\Rightarrow\; \lambda \in \Omega_- \Rightarrow \operatorname{Re}\lambda < -(\beta+\zeta), \text{ and} \tag{6.40}$$

$$\operatorname{Re}\lambda > 0 \;\Rightarrow\; \lambda \in \Omega_+ \Rightarrow \operatorname{Re}\lambda > \zeta. \tag{6.41}$$

So for each c satisfying (6.39), we can write

$$e^{sB^c} = \frac{1}{2\pi i}\int_{\partial\Omega_-} e^{sz}(zI - B^c)^{-1}\,dz + \frac{1}{2\pi i}\int_{\partial\Omega_+} e^{sz}(zI - B^c)^{-1}\,dz. \tag{6.42}$$

Now

$$\mathbb{R}^{2N} = E^c_- \oplus E^c_+, \tag{6.43}$$

where E^c_- and E^c_+ are subspaces of $\mathbb{R}^{2N}$, invariant under B^c, that are spanned by the generalised eigenvectors of B^c that correspond to eigenvalues of B^c in Ω_- and Ω_+ respectively. Denote the corresponding projections by $\pi^c_- : \mathbb{R}^{2N} \to E^c_-$ and $\pi^c_+ : \mathbb{R}^{2N} \to E^c_+$. Then

$$\pi^c_- + \pi^c_+ = I, \tag{6.44}$$

$$\pi^c_-\pi^c_+ = \pi^c_+\pi^c_- = 0, \tag{6.45}$$

$$\text{and } \; {\pi^c_-}^2 = \pi^c_-, \quad {\pi^c_+}^2 = \pi^c_-. \tag{6.46}$$

Kato [26] gives the integral representations

$$\pi^c_- = \frac{1}{2\pi i}\int_{\partial\Omega_-} (zI - B^c)^{-1}\,dz, \quad \pi^c_+ = \frac{1}{2\pi i}\int_{\partial\Omega_+} (zI - B^c)^{-1}\,dz. \tag{6.47}$$

Now by (6.44),

$$e^{sB^c} = U^c_-(s) + U^c_+(s), \quad s \in \mathbb{R}, \tag{6.48}$$

where

$$U^c_-(s) := e^{sB^c}\pi^c_- \text{ and } U^c_+(s) := e^{sB^c}\pi^c_+. \tag{6.49}$$

Consider the first term on the right-hand side of (6.49). Cauchy's Theorem and (6.42) yield that on $\mathbb{R}^{2N}$,

$$U^c_-(s) = \left\{\int_{\partial\Omega_-} e^{sz}(zI - B^c)^{-1}\,dz\right\}\pi^c_-. \tag{6.50}$$

Clearly, $\{B^c : |c| \leq \gamma\}$ is compact in the usual topology of $2N \times 2N$ matrices.

Hence (6.47) yields that there exist $K_1, K_2 > 0$, independent of c satisfying (6.39), such that

$$\|\pi_-^c\| \leq K_1, \tag{6.51}$$

and using (6.40),

$$\left\| \int_{\partial\Omega_-} e^{sz}(zI - B^c)^{-1}\, dz \right\| \leq K_2 e^{-s(\beta+\zeta)}, \quad s \geq 0. \tag{6.52}$$

Hence

$$\left\| U_-^c(s) \right\| \leq K e^{-s(\beta+\zeta)}, \quad s \geq 0, \tag{6.53}$$

where $K = K_1K_2 > 0$. Similarly,

$$\left\| U_+^c(s) \right\| \leq K e^{s\zeta}, \quad s \leq 0 \tag{6.54}$$

where $K > 0$ is chosen larger if necessary.

For reasons to be made clear later, we choose $\epsilon > 0$ so that $\dfrac{2\epsilon K}{\zeta} < \dfrac{1}{2}$, and then choose $\eta > 0$ so that (6.38) holds, and $\nu > 0$ such that $\nu < \min\{\eta, \frac{\eta}{2K}\}$. Since $w = u + \psi$ is monotone, Theorem 6.1.2 yields the existence of $N_2 > 0$, independent of w, c and σ, such that

$$\|w''\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < N_2. \tag{6.55}$$

So for each $i \in \{1, \ldots, N\}$, Lemma 6.1.1 and the monotonicity of w yield

$$\begin{aligned} (w_i'(s))^2 = -2\int_s^\infty w_i'(t)w_i''(t)\, dt &\leq 2\int_s^\infty |w_i'(t)||w_i''(t)|\, dt \\ &\leq 2N_2 \int_s^\infty w_i'(t)\, dt \\ &= 2N_2[T_i - w_i(s)]. \end{aligned} \tag{6.56}$$

Thus by (6.34) and (6.56), there exists $\delta > 0$, independent of w, c and σ such that

$$\|w(s) - T\| \leq \delta \Rightarrow \|p(s) - p_0\| \leq \nu, \quad s \in \mathbb{R}. \tag{6.57}$$

By condition **(f3)** and the Inverse Function Theorem, T is an isolated zero of f. Therefore $\delta > 0$ can be chosen smaller if necessary so that (6.32) is satisfied.

Now let $s_0 \in \mathbb{R}$ be such that $\|w(s_0) - T\| = \delta$; that $\|w(s) - T\| \leq \delta$ for

$s \geq s_0$ then follows since $w'(s) > 0$ for each s. Pre-multiplying (6.35) by e^{-sB^c} and integrating from s_0 to s leads to

$$p(s) - p_0 = e^{(s-s_0)B^c}(p(s_0) - p_0) + \int_{s_0}^{s} e^{(s-t)B^c} H_\sigma(t, p(t) - p_0)\, dt, \tag{6.58}$$

which, using (6.48), becomes

$$\begin{aligned} p(s) - p_0 &= U_-^c(s - s_0)(p(s_0) - p_0) + U_+^c(s - s_0)(p(s_0) - p_0) \\ &+ \int_{s_0}^{s} U_-^c(s-t) H_\sigma(t, p(t) - p_0)\, dt + \int_{s_0}^{s} U_+^c(s-t) H_\sigma(t, p(t) - p_0)\, dt. \end{aligned} \tag{6.59}$$

(6.57) and (6.38) imply that $H_\sigma(s, p(s) - p_0)$ is bounded independently of $s \geq s_0$. This, together with estimate (6.54) and Lebesgue's Dominated Convergence Theorem, gives that the existence of the integrals

$$\int_{s_0}^{\infty} U_+^c(s-t) H_\sigma(t, p(t) - p_0)\, dt, \quad \int_{s}^{\infty} U_+^c(s-t) H_\sigma(t, p(t) - p_0)\, dt.$$

Hence (6.59) can be rewritten as

$$\begin{aligned} p(s) - p_0 &= U_-^c(p(s_0) - p_0) + U_+^c(s - s_0)k_0 \\ &+ \int_{s_0}^{s} U_-^c(s-t) H_\sigma(t, p(t) - p_0)\, dt - \int_{s}^{\infty} U_+^c(s-t) H_\sigma(t, p(t) - p_0)\, dt, \end{aligned} \tag{6.60}$$

where $k_0 \in \mathbb{R}^{2N}$ is

$$k_0 := p(s_0) - p_0 + \int_{s_0}^{\infty} U_+^c(s_0 - t) H_\sigma(t, p(t) - p_0)\, dt. \tag{6.61}$$

Estimates (6.53) and (6.54) and the fact that $H_\sigma(s, p(s) - p_0)$ is bounded independently of $s \geq s_0$ imply that the terms on the right of (6.60) other than $U_+^c(s - s_0)k_0$ are bounded independently of $s \geq s_0$. The same is true of the left-hand side of (6.60), by (6.57). It follows that $U_+^c(s - s_0)k_0$ is bounded independently of $s \geq s_0$. But for $s \geq s_0$,

$$U_+^c(s - s_0)k_0 = e^{(s-s_0)B^c}\pi_+^c k_0 = e^{(s-s_0)B^c}\Big|_{E_+^c}\pi_+^c k_0 = e^{(s-s_0)B^c|_{E_+^c}}\pi_+^c k_0, \tag{6.62}$$

where $|_{E_+^c}$ denotes restriction to the subspace E_+^c. Since $B^c|_{E_+^c}$ is a source, it follows from [24] that there exists $L, a > 0$ such that for $s \geq s_0$,

$$\|U_+^c(s-s_0)k_0\| \geq Le^{(s-s_0)a}\|\pi_+^c k_0\|. \tag{6.63}$$

The boundedness of $U_+^c(s-s_0)k_0$ as s approaches infinity thus yields that $\pi_+^c k_0 = 0$, and so $U_+^c(s-s_0)k_0 = 0$ for each $s \geq s_0$. Hence $y(s) := p(s), s \geq s_0$ is a solution of

$$\begin{aligned} y(s) - p_0 &= U_-^c(s-s_0)(p(s_0) - p_0) \\ &+ \int_{s_0}^{s} U_-^c(s-t)H_\sigma(t, y(t) - p_0)\, dt - \int_s^\infty U_+^c(s-t)H_\sigma(t, y(t) - p_0)\, dt, \end{aligned} \tag{6.64}$$

and

$$\|y(s) - p_0\| \leq \eta \text{ for } s \geq s_0. \tag{6.65}$$

To show that $p(s)$ satisfies estimate (6.30), we will first construct *a* solution to (6.64) by Picard's method, which satisfies (6.30). That the estimate holds for p will then be shown by proving that the solution to (6.64) is unique.
So let $y_0(s) = p_0, s \geq s_0$, and for each $n \geq 0$, define

$$\begin{aligned} y_{n+1}(s) &= p_0 + U_-^c(s-s_0)(p(s_0) - p_0) \\ &+ \int_{s_0}^{s} U_-^c(s-t)H_\sigma(t, y_n(t) - p_0)\, dt - \int_s^\infty U_+^c(s-t)H_\sigma(t, y_n(t) - p_0)\, dt \end{aligned} \tag{6.66}$$

for $s \geq s_0$. We will verify by induction that for each $n \geq 0$,

$$\|y_{n+1}(s) - y_n(s)\| \leq \frac{K\|p(s_0) - p_0\|e^{-\beta(s-s_0)}}{2^n}, \tag{6.67}$$

$$\|y_{n+1}(s) - p_0\| \leq \eta, \tag{6.68}$$

for $s \geq s_0$. That (6.67) and (6.68) hold for $n = 0$ follows from (6.37), estimate (6.53) and the fact that $2K\nu < \eta$. Now let $k \in \mathbb{N}$ and suppose that (6.67) holds for $0 \leq n \leq k-1$. Then

$$\begin{aligned} \|y_{k+1}(s) - y_k(s)\| &= \Big\| \int_{s_0}^{s} U_-^c(s-t)\{H_\sigma(t, y_k(t) - p_0) - H_\sigma(t, y_{k-1}(t) - p_0)\}\, dt \\ &\quad - \int_s^\infty U_+^c(s-t)\{H_\sigma(t, y_k(t) - p_0) - H_\sigma(t, y_{k-1}(t) - p_0)\}\, dt \Big\| \\ &\leq \epsilon \int_{s_0}^{s} Ke^{-(\beta+\zeta)(s-t)}\|(y_k(t) - y_{k-1}(t)\|\, dt \end{aligned}$$

$$
\begin{aligned}
& +\epsilon \int_s^\infty K e^{\zeta(s-t)} \|y_k(t) - y_{k-1}(t)\| \, dt \\
\leq \;& \frac{\epsilon K^2 e^{-\beta(s-s_0)}}{\zeta 2^{k-1}} \|p(s_0) - p_0\| \left\{1 - e^{-\zeta(s-s_0)}\right\} + \frac{\epsilon K^2 e^{-\beta(s-s_0)}}{(\zeta+\beta)2^{k-1}} \\
\leq \;& \left(\frac{2\epsilon K}{\zeta}\right)\left(\frac{K\|p(s_0) - p_0\| e^{-\beta(s-s_0)}}{2^{k-1}}\right) \\
\leq \;& \frac{K\|p(s_0) - p_0\| e^{-\beta(s-s_0)}}{2^k}
\end{aligned}
$$

using the inductive hypothesis, (6.38), and the fact that $\frac{2\epsilon K}{\zeta} < \frac{1}{2}$. Furthermore,

$$
\begin{aligned}
\|y_{k+1}(s) - p_0\| \;&\leq\; \sum_{n=0}^{k} \|y_{n+1}(s) - y_n(s)\| \\
&\leq\; K\|p(s_0) - p_0\| e^{-\beta(s-s_0)} \sum_{n=0}^{k} 2^{-n} \\
&\leq\; 2K\epsilon \;<\; \nu. \qquad (6.69)
\end{aligned}
$$

It follows by induction that (6.67) holds for all $n \geq 0$. The Weierstrass M-test and (6.67) imply that

$$
y_0(s) + \sum_{n=0}^{\infty} [y_{n+1}(s) - y_n(s)] =: y(s) \qquad (6.70)
$$

converges uniformly on $[s_0, \infty)$; that is,

$$
y(s) = \lim_{n\to\infty} y_n(s) \qquad (6.71)
$$

exists uniformly on $[s_0, \infty)$. Since $\|y_n(s) - p_0\| \leq \eta$ for each $n \geq 0, s \geq s_0$ and estimates (6.53) and (6.54) hold, we can pass to the limit as $n \to \infty$ in (6.66) using Lebesgue's Dominated Convergence Theorem to obtain that y satisfies (6.64) and (6.65). Moreover, (6.70) and (6.67) yield that for $s \geq s_0$,

$$
\|y(s) - p_0\| \leq 2K\|p(s_0) - p_0\| e^{-\beta(s-s_0)} = \kappa e^{-\beta(s-s_0)}, \qquad (6.72)
$$

where $\kappa := 2K\nu$.

It remains to show that the solution to (6.64) and (6.65) is unique. Let y_1, y_2

be solutions of (6.64), (6.65), and let

$$M := \sup_{s \geq s_0} \|y_1(s) - y_2(s)\|.$$

Then by (6.64), (6.53), (6.65) and (6.37), for $s \geq s_0$,

$$\|y_1(s)-y_2(s)\| \leq \epsilon K e^{-\zeta s} \int_{s_0}^{s} e^{\zeta t}\|y_1(s)-y_2(s)\|\,dt + \epsilon K e^{\zeta s} \int_{s}^{\infty} e^{-\zeta t}\|y_1(s)-y_2(s)\|\,dt$$

and hence

$$M \leq \frac{2\epsilon K}{\zeta} M \leq \frac{M}{2}.$$

Whence $M = 0$. The result follows.

$\square$

Recall here Theorem 2.3.1 from Chapter 2. This gives a necessary condition for the existence of monotone directions to which a solution tends monotonically. The importance of this theorem for the gradient-dependent travelling-wave problem lies in the following application, which is the motivation for hypothesis **(f4)** on the function f.

Lemma 6.1.7 *Let $w_0 \in \mathbb{R}^N$ be such that $f(w_0) = 0$, $df[w_0] \in P^{N\times N}$ and $\mu_{PF}(df[w_0]) > 0$. Then for a fixed $c \in \mathbb{R}$, there cannot exist two functions $w_1, w_2 \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $\sigma_1, \sigma_2 \in C^1(\mathbb{R}, [0,1])$ such that w_i satisfies (2.7) and (2.8), with $\sigma = \sigma_i$ $(i = 1, 2)$, and*

$$w_1(s) \to w_0 \quad as \quad s \to -\infty, \quad w_1'(s) \geq 0 \textit{ for sufficiently large } -s, \tag{6.73}$$

$$w_2(s) \to w_0 \quad as \quad s \to \infty, \quad w_2'(s) \geq 0 \textit{ for sufficiently large } s. \tag{6.74}$$

Proof. Suppose that there is $c \in \mathbb{R}$ such that functions w_1, w_2 as in the statement of the lemma exist. Then it follows from Theorem 2.3.1 that there are real numbers $\lambda_1 < 0$, $\lambda_2 > 0$ and vectors $q_1, q_2 > 0$, such that

$$(\lambda_i^2 A + \lambda_i c I + df[w_0])q_i = 0, \quad i = 1, 2. \tag{6.75}$$

But it is shown in Lemma 3.2.4 that for c fixed, the Perron-Frobenius eigenvalue of $M(\lambda, c) = \lambda^2 A + c\lambda I + df[w_0]$ is a strictly convex function of λ. Thus since $\mu_{PF}(df[w_0]) = \mu_{PF}(M(0,c)) > 0$, there cannot exist both $\lambda_1 < 0$ and $\lambda_2 > 0$ such that the Perron-Frobenius eigenvalue of $\lambda_i^2 A + \lambda_i cI + df[w_0]$ is zero. This contradicts (6.75) because the only positive eigenvector of $M \in P^{N\times N}$ is that corresponding to $\mu_{PF}(M)$. The result follows.

□

Our final preliminary result shows that the assumption of the local monotonicity of f (condition **(f1)**) forces monotone solutions of the system (2.7) to be *strictly* monotone.

Lemma 6.1.8 *Let $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (2.7) for some $\sigma \in C^1(\mathbb{R}, [0,1])$. Suppose that for each $s \in \mathbb{R}$,*

$$w'(s) \geq 0. \tag{6.76}$$

Then either there exists $w_0 \in \mathbb{R}^N$ such that

$$\begin{aligned} w(s) &\equiv w_0 \text{ for each } s \in \mathbb{R}, \text{ or} &(6.77)\\ w'(s) &> 0 \text{ for each } s \in \mathbb{R}. &(6.78)\end{aligned}$$

Proof. Suppose that there exists $s_0 \in \mathbb{R}$ for which $w'(s_0)$ has a zero component; without loss of generality, say $w_1'(s_0) = 0$. Consider the first equation of (2.7),

$$A_1 w_1''(s) + cw_1'(s) + \sigma(s) G_1(w(s), w'(s)) w_1'(s) + f_1(w(s)) = 0, \quad s \in \mathbb{R}. \tag{6.79}$$

If $f_1(w(s_0)) \neq 0$, (6.79) implies that $w_1''(s_0) \neq 0$, which says that w_1 has either a maximum or a minimum at s_0. Since this contradicts (6.76), $f_1(w(s_0)) = 0$, so $w_1''(s_0) = 0$. Now differentiation of (6.79) at s_0 gives

$$A_1 w_1'''(s_0) = -\sum_{i=2}^{N} \frac{\partial f_1}{\partial w_i}(w(s_0)) w_i'(s_0), \tag{6.80}$$

since $w_1'(s_0) = w_1''(s_0) = 0$.

Suppose first that $w'(s_0) \neq 0$. Then **(f1)** and (6.76) yield that $w_1'''(s_0) < 0$, which contradicts (6.76) since $w_1''(s_0) = 0$. Hence $w'(s_0) = 0$. Since $w_i'(s_0) = 0$ for all i, the above argument then gives that $w_i''(s_0) = 0$ and hence $f(w(s_0)) = 0$. So uniqueness of the solution v to the initial value problem (2.7) with $v(s_0) = w(s_0)$, $v'(s_0) = w'(s_0)$ yields that $w(s) \equiv w(s_0)$, $s \in \mathbb{R}$. Hence (6.77) holds, and the result follows.

□

6.2 Estimates for the approximate system that are independent of R

Consider now the system

$$A(u'' + \psi'') + c[u' + \psi'] + \tau\sigma_R G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0, \quad (6.81)$$

where $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$, $\tau \in [0, 1]$, $c \in \mathbb{R}$, $R > 0$ and σ_R is as in section 5.2.1. As usual, f satisfies **(f1)-(f4)** and G satisfies **(G1)-(G3)**. This system is a specific case of (6.2), where here $\sigma = \tau\sigma_R$. Hence the results of section 6.1 apply to (6.81). We will now prove *a priori* lower bounds on $w_i'(s)$, $i \in \{1, \ldots, N\}$, when $w(s)$ is outside the δ-neighbourhoods of S and T constructed in Theorem 6.1.6 and $u = w - \psi$ satisfies (6.81).

Theorem 6.2.1 *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (6.81) for some $c \in \mathbb{R}$, $\tau \in [0, 1]$ and $R > 0$, and suppose that $u'(s) + \psi'(s) > 0$ for all $s \in \mathbb{R}$. Then there exists $\chi > 0$, independent of u, c, τ, and R, such that for $w = u + \psi$,*

$$w_i'(s) > \chi, \quad i = 1, \ldots, N, \quad (6.82)$$

when $\|w(s) - S\| \geq \delta$ and $\|w(s) - T\| \geq \delta$, where δ is as in Theorem 6.1.6.

Proof. Suppose that the result is false. Then there exist sequences $\{u^k\} \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$, $\{\tau_k\} \subset [0, 1]$, $\{c_k\} \subset \mathbb{R}$, $\{R_k\} \subset (0, \infty)$, $\{s_k\} \subset \mathbb{R}$ and $\{i^k\} \subset \{1, \ldots, N\}$ such that u^k satisfies (6.81) with $c = c_k$, $\tau = \tau_k$ and $R = R_k$,

$$\|w^k(s_k) - S\| \geq \delta \text{ and } \|w^k(s_k) - T\| \geq \delta, \quad (6.83)$$

and

$$0 < \left(w^k_{i_k}\right)'(s_k) < \frac{1}{k}, \tag{6.84}$$

where $w^k := u^k + \psi$. Since Theorem 6.1.5 gives that $\{c_k\}$ is bounded, there is a subsequence of $\{u^k\}$, $\left\{u^{k(j)}\right\}_{j=1}^{\infty}$ and constants $\tau_0 \in [0,1]$, $c_0 \in \mathbb{R}$ and $i_0 \in \{1,\dots,N\}$ such that $c_{k(j)} \to c_0$, $\tau_{k(j)} \to \tau_0$ as $k \to \infty$ and $i_{k(j)} = i_0$ for each $j \in \mathbb{N}$. Without loss of generality, we suppose henceforth that $i_0 = 1$ and write k for $k(j)$.

Now let $\{t_k\} \subset \mathbb{R}$ be such that $\|w^k(t_k) - T\| = \delta$, and define

$$v^k(s) = w^k(s + t_k), \quad s \in \mathbb{R}, \tag{6.85}$$

so that for $s \in \mathbb{R}$,

$$A(v^k)''(s) + c_k(v^k)'(s) + \tau_k\sigma_{R_k}(s+t_k)G(v^k(s),(v^k)'(s))(v^k)'(s) + f(v^k(s)) = 0 \tag{6.86}$$

and

$$\|v^k(0) - T\| = \delta. \tag{6.87}$$

Also, by (6.85), (6.83) and (6.84),

$$\left(v^k_1\right)'(s_k - t_k) = \left(w^k_1\right)'(s_k) \to 0 \text{ as } k \to \infty, \tag{6.88}$$

and

$$\|v^k(s_k - t_k) - S\| \geq \delta, \ \ \|v^k(s_k - t_k) - T\| \geq \delta. \tag{6.89}$$

Since w^k is a monotone solution connecting S and T for each k,

$$\|w^k\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq M \text{ for each } k,$$

where $M > 0$ is as in (6.23), and is independent of w^k, c_k, τ_k and R_k. Hence by Theorem 6.1.2, there exist $N_1, N_2 > 0$, independent of w^k, c_k, τ_k and R_k, such that

$$\|(w^k)'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq N_1, \ \ \|(w^k)''\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq N_2, \tag{6.90}$$

and hence

$$\|(v^k)'(s)\| \leq N_1, \quad \|(v^k)''(s)\| \leq N_2 \text{ for all } s \in \mathbb{R}. \tag{6.91}$$

We aim to pass to a limit in (6.86) as $k \to \infty$ through a subsequence. Let $r > 0$ and define $Y_r := [-r, r]$. Using (6.91) and arguing using the Arzela-Ascoli Theorem as in the proof of Theorem 2.3.1, it follows that there is a subsequence of $\{v^k\}$, say $\{v^j\}$, and $u_1, u_2 \in C(Y_r, \mathbb{R}^N)$ such that $v^j \to u_1$ and $(v^j)' \to u_2$ as $j \to \infty$.

Now define

$$\sigma^k(s) = \sigma_{R_k}(s + t_k), \quad s \in \mathbb{R}, \tag{6.92}$$

for each $k \in \mathbb{N}$. Recall from the definition of σ_R in section 5.2.1 that $\{\sigma_R\}$ and $\{\sigma_R'\}$ are each uniformly bounded and equicontinuous as families parametrised by R. Hence $\{\sigma^k\}$ and $\{\left(\sigma^k\right)'\}$ are each uniformly bounded and equicontinuous as families parametrised by k. So given $r > 0$, there is a subsequence such that σ^j converges in $C^1([-r, r], [0, 1])$. A diagonal sequence argument gives that there exists $\sigma \in C^1(\mathbb{R}, [0, 1])$ such that a further subsequence $\sigma^j \to \sigma$ in $C^1([-r, r], [0, 1])$ for each $r > 0$. So for each $s \in \mathbb{R}$,

$$\sigma_{R_j}(s + t_j) = \sigma^j(s) \to \sigma(s) \text{ as } j \to \infty. \tag{6.93}$$

Since G and f are continuous and v^j, $(v^j)'$ are uniformly convergent on Y_r, it follows from (6.86) and (6.93) that $(v^j)''$ converges uniformly on Y_r. Hence $\{v^j\}$ is Cauchy in $C^2(Y_r, \mathbb{R}^N)$, and thus there exists $v \in C^2(Y_r, \mathbb{R}^N)$ such that $v^j \to v$ in $C^2(Y_r, \mathbb{R}^N)$ as $j \to \infty$. A diagonal subsequence argument now shows that $\{v^k\}$ has a subsequence that converges in $C^2(Y_r, \mathbb{R}^N)$ for each $r > 0$ to a limit $v \in C^2(\mathbb{R}, \mathbb{R}^N)$.

Passing to the limit in (6.86) as $j \to \infty$ along this subsequence yields that

$$Av''(s) + c_0 v'(s) + \tau_0 \sigma(s) G(v(s), v'(s)) v'(s) + f(v(s)) = 0, \quad s \in \mathbb{R}. \tag{6.94}$$

By (6.87),

$$\|v(0) - T\| = \delta, \tag{6.95}$$

and since $S \le v^k(s) \le T$ for each $k \in \mathbb{N}$, $s \in \mathbb{R}$,

$$S \le v(s) \le T, \quad s \in \mathbb{R}. \tag{6.96}$$

Also, since $\left(v^k\right)'(s) > 0$ for each $k \in \mathbb{N}$, $s \in \mathbb{R}$,

$$v'(s) \ge 0, \quad s \in \mathbb{R}, \tag{6.97}$$

so it follows from (6.96) that there exist $p, q \in \mathbb{R}^N$, with $S \le p \le q \le T$ such that

$$v(s) \to p \text{ as } s \to -\infty \text{ and } v(s) \to q \text{ as } s \to \infty. \tag{6.98}$$

Moreover, since $\|v'(s)\| \le N_1, \|v''(s)\| \le N_2$ by (6.91), and hence $v'''(s)$ is uniformly bounded by (6.94), it follows using Landau's inequality that $v'(s), v''(s) \to 0$ as $|s| \to \infty$. So using (6.96), we find that $f(p) = f(q) = 0$. It thus follows from (6.95), (6.96) and (6.32) that $q = T$.

Next suppose, for contradiction, that $p = S$. Let $x \in \mathbb{R}$ be such that $\|v(x) - S\| = \frac{\delta}{2}$. Then since $v^j(x) \to v(x)$ as $j \to \infty$, there exists $j_0 \in \mathbb{N}$ such that $j \ge j_0 \Rightarrow \|v^j(x) - S\| \le \delta$. Hence $x < s_j - t_j < 0$ for $j \ge j_0$ (recall (6.89)). So $\{s_j - t_j\}_{j=j_0}^{\infty}$ is contained in a finite interval in $\mathbb{R}$, $[x, 0]$, thus has a convergent subsequence, say $s_j - t_j \to x_0 \in [x, 0]$. Since $v^j(s) \to v(s)$ uniformly for $s \in [x, 0]$ and $v_1^j(s_j - t_j) \to 0$ as $j \to \infty$, it follows that $v_1'(x_0) = 0$. But this contradicts Lemma 6.1.8 since v satisfies (6.94) and (6.97) holds. So $p \ne S$.

Recall from condition **(f4)** that there are only a finite number of zeros w_0 of f with $S \le w_0 \le T$. So we can choose $\epsilon > 0$ sufficiently small that

$$f(w_0) = 0, \ \|w_0 - p\| \le \epsilon \Rightarrow w_0 = p. \tag{6.99}$$

Moreover, Lemma 4.0.11 implies that

$$f(w_0) = 0, w_{0i} = p_i \text{ for some } i \in \{1, \dots, N\} \Rightarrow w_0 = p, \tag{6.100}$$

so ϵ can be chosen, smaller if necessary, so that

$$f(w_0) = 0, \ |w_{0_i} - p_i| \le \epsilon \text{ for some } i \in \{1, \dots, N\} \Rightarrow w_0 = p. \tag{6.101}$$

Now let $E := (\epsilon, \dots, \epsilon)$. For $n \in \mathbb{N}$, choose $x_n \in \mathbb{R}$ such that $p \le v(x_n) \le p + \frac{E}{n}$. Since $v^k(x_n) \to v(x_n)$ as $k \to \infty$, we can choose k_n ($> k_{n-1}$) such that $p \le v^k(x_n) \le p + \frac{E}{n}$ also. Since $v^{k_n}(s) \to S$ as $s \to -\infty$ monotonically, there exists unique $\alpha_{k_n} \in \mathbb{R}$ such that $\|v^{k_n}(\alpha_{k_n}) - p\| = \epsilon$ and $\|v^{k_n}(s) - p\| \ge \epsilon$ for $s \le \alpha_{k_n}$. For each $n \in \mathbb{N}$, define

$$\widetilde{v^{k_n}}(s) = v^{k_n}(s + \alpha_{k_n}), \quad s \in \mathbb{R}. \tag{6.102}$$

Then $\|\widetilde{v^{k_n}}(0) - p\| = \epsilon$ for each n, and

$$S \le \widetilde{v^{k_n}}(0) \le p + \frac{E}{n}. \tag{6.103}$$

Arguing as in the construction of the function v above, we obtain the existence of $\tilde{v} \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $\tilde{\sigma} \in C^1(\mathbb{R}, [0,1])$ such that (for a subsequence) $\widetilde{v^{k_n}} \to \tilde{v}$ uniformly on compact subsets of $\mathbb{R}$,

$$\|\tilde{v}(0) - p\| = \epsilon, \quad S \le \tilde{v}(0) \le p, \quad \tilde{v}'(s) \ge 0, \; s \in \mathbb{R}, \tag{6.104}$$

and

$$A\tilde{v}'' + c_0\tilde{v}' + \tau_0\tilde{\sigma}G(\tilde{v}, \tilde{v}')\tilde{v}' + f(\tilde{v}) = 0. \tag{6.105}$$

As above, there exist $p', q' \in \mathbb{R}^N$ such that

$$S \le p' \le q' \le T, \quad \text{and}$$

$$\tilde{v}(s) \to p' \;\text{ as }\; s \to -\infty \;\text{ and }\; \tilde{v}(s) \to q' \;\text{ as }\; s \to \infty.$$

Now by condition **(f4)**, $\mu_{PF}(df[w_0]) > 0$ for each w_0, $S < w_0 < T$ with $f(w_0) = 0$. Theorem 4.1.2 implies the existence of a zero w_0 of f such that $\mu_{PF}(df[w_0]) < 0$ in the order interval between two zeros whose Fréchet derivatives have positive Perron-Frobenius eigenvalues. Hence there is no zero of f such that $S < w_0 < p$ or $p < w_0 < T$. Thus $p' = S$ since $\|\tilde{v}(0) - p\| = \epsilon$. Also, (6.101) yields that $q' \in \{p, T\}$. If $q' = T$, then as earlier in this proof, the points $s_{k_n} - t_{k_n} - \alpha_{k_n}$ at which $\left(v_1^{k_n}\right)'$ tends to zero are contained in a finite interval in $\mathbb{R}$. So there exists $z_0 \in \mathbb{R}$ such that $(\tilde{v_1})'(z_0) = 0$, which contradicts Lemma 6.1.8 since (6.104) and (6.105) hold. Hence $q' = T$.

Thus we have $\tilde{v}$ satisfying (6.105) with $\tilde{v}(s) \leq p, \quad s \in \mathbb{R}$, $\tilde{v}(s) \to p$ as $s \to \infty$, and v satisfying (6.94) with $v(s) \geq p, \quad s \in \mathbb{R}$, $v(s) \to p$ as $s \to -\infty$. But since $\mu_{PF}(df[p]) > 0$, this contradicts Lemma 6.1.7. The result follows.

$\square$

Let $\mathcal{W}_r \quad (r \geq 0)$ denote the set of all monotone functions $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $w = u + \psi$ where $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies (6.2) for some $c \in \mathbb{R}$ and some $\sigma \in C^1(\mathbb{R}, [0, 1])$ and

$$w_1(\hat{s}) = \frac{1}{2}(T_1 + S_1) \text{ for } |\hat{s}| \leq r. \tag{6.106}$$

(Here w_1 denotes the first component of w). The penultimate step in bounding $\mathcal{M}_R$ (6.1) in the space $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ independently of $R > 0$ is the following.

Lemma 6.2.2 *There exists constant $C_r > 0$ such that for any $w \in \mathcal{W}_r$,*

$$\|w - \psi\|_{W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)} \leq C_r. \tag{6.107}$$

Proof. Let $w \in \mathcal{W}_r$. Let s_0, t_0 and $\delta > 0$ be as in Theorem 6.1.6. Since $\delta < \frac{1}{2} \min_{1 \leq i \leq N} |T_i - S_i|$,

$$s_0 < \hat{s} < t_0, \quad |\hat{s}| \leq r. \tag{6.108}$$

Now let $\chi > 0$ be as in Theorem 6.2.1, so that $w_i'(s) \geq \chi$ for each $i \in \{1, \ldots, N\}$ whenever $\|w(s) - S\| \geq \delta$ and $\|w(s) - T\| \geq \delta$. Without loss of generality, it can be assumed that $\chi < 1$ so that

$$0 < t_0 - s_0 < \frac{1}{\chi}, \tag{6.109}$$

which together with (6.108) yields that

$$s_0 > -r - \frac{1}{\chi} \text{ and } t_0 < r + \frac{1}{\chi}. \tag{6.110}$$

By Minkowski's inequality,

$$\left(\int_{-r-\frac{1}{\chi}}^{r+\frac{1}{\chi}} \|w - \psi\|^2 \mu \, ds \right)^{\frac{1}{2}} \leq \left(\int_{-r-\frac{1}{\chi}}^{r+\frac{1}{\chi}} \|w\|^2 \mu \, ds \right)^{\frac{1}{2}} + \left(\int_{-r-\frac{1}{\chi}}^{r+\frac{1}{\chi}} \|\psi\|^2 \mu \, ds \right)^{\frac{1}{2}}$$

$$\leq \Upsilon(r,\chi)\|T-S\| + \left(\int_{-r-\frac{1}{\chi}}^{r+\frac{1}{\chi}} \|\psi\|^2 \mu\, ds\right)^{\frac{1}{2}}. \tag{6.111}$$

where $\Upsilon(r,\chi) = \left(2(r+\frac{1}{\chi})(1+(r+\frac{1}{\chi})^2)\right)^{\frac{1}{2}}$. Since $\chi < 1$, $\psi \equiv S$ when $s \leq -r-\frac{1}{\chi}$ and $\psi(s) = T$ when $s \geq r+\frac{1}{\chi}$. So (6.110), the choice of s_0 and t_0, and the monotonicity of w give that

$$\|w(s)-\psi(s)\| \leq \delta \text{ whenever } s \leq -r-\frac{1}{\chi} \text{ or } s \geq r+\frac{1}{\chi}. \tag{6.112}$$

Thus by the exponential estimates of Theorem 6.1.6,

$$\begin{aligned}
\int_{r+\frac{1}{\chi}}^{\infty} \|w(s)-\psi(s)\|^2 \mu(s)\, ds &= \int_{r+\frac{1}{\chi}}^{\infty} \|w(s)-S\|^2 \mu(s)\, ds \\
&\leq \kappa^2 \int_{r+\frac{1}{\chi}}^{\infty} e^{-2\beta(s-t_0)} \mu(s)\, ds \\
&\leq \kappa^2 e^{2\beta(r+\frac{1}{\chi})} \int_{r+\frac{1}{\chi}}^{\infty} e^{-2\beta s}(1+s^2)\, ds, \qquad (6.113)
\end{aligned}$$

using $\mu(s) := 1+s^2$ and the fact that $t_0 < r+\frac{1}{\chi}$. Since the right-hand side of (6.113) is finite and independent of $w \in \mathcal{W}_r$, it follows using the analogous estimate at S and (6.111) that there exists $M_1 > 0$ such that for every $w \in \mathcal{W}_r$,

$$\left(\int_{\mathbb{R}} \|w-\psi\|^2 \mu\, ds\right)^{\frac{1}{2}} \leq M_1.$$

Using estimate (6.24) (as guaranteed by Theorem 6.1.2) and the exponential estimates for w' in Theorem 6.1.6, it can be argued similarly that there exists $M_2 > 0$ such that for every $w \in \mathcal{W}_r$,

$$\left(\int_{\mathbb{R}} \|w'-\psi'\|^2 \mu\, ds\right) \leq M_2.$$

The lemma follows.

□

Up to this point in Chapter 6, the velocity c has been treated as a constant. We now reintroduce the representation of c as a functional to obtain the uniform Sobolev space bound on $\mathcal{M}_R$.

Theorem 6.2.3 *There exists $C > 0$ such that if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies*

$$A(u'' + \psi'') + c(u)(u' + \psi') + \tau\sigma_R G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0, \quad (6.114)$$

for some $\tau \in [0, 1]$, $R > 0$, where the functional $c(\cdot)$ is as defined in (5.24) and (5.25), and

$$u'(s) + \psi'(s) > 0 \text{ for each } s \in \mathbb{R}, \quad (6.115)$$

then

$$\|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \leq C. \quad (6.116)$$

Proof. Let u, w be as in the statement of the theorem. Denote by $\hat{s}$ the solution of the equation $w_1(\hat{s}) = \frac{1}{2}(T_1 + S_1)$. It will be proved that there exists $r > 0$, independent of u, τ, R satisfying (6.114), (6.115), such that

$$|\hat{s}| \leq r.$$

If not, then there exists a sequence $\{u^k\} \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfying (6.114), (6.115) for some τ_k, R_k, and a sequence $\{s_k\} \subset \mathbb{R}$ with $|s_k| \to \infty$ as $k \to \infty$, such that

$$w^k_1(s_k) = u^k_1(s_k) + \psi_1(s_k) = \frac{1}{2}(T_1 + S_1). \quad (6.117)$$

We will show that the set of the corresponding velocities $\{c(u^k)\}$ is unbounded, which will contradict Theorem 6.1.5. Define

$$v^k(s) := w^k(s + s_k) - \psi(s), \quad s \in \mathbb{R}, \quad (6.118)$$

so that

$$v^k_1(0) + \psi_1(0) = w^k_1(s_k) = \frac{1}{2}(T_1 + S_1) \quad (6.119)$$

and hence $v^k + \psi \in \mathcal{W}_0$, where $\mathcal{W}_0$ is as in Lemma 6.2.2. Thus there exists $C_0 > 0$, independent of k, such that

$$\|v^k\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \leq C_0. \quad (6.120)$$

Now by (5.24) and (6.118),

$$\begin{aligned}\rho(u^k)^2 &= \int_{\mathbb{R}} \|w^k(s) - T\|^2 \phi(s)\, ds \\ &= \int_{\mathbb{R}} \|v^k(s) + \psi(s) - T\|^2 \phi(s + s_k)\, ds. \qquad (6.121)\end{aligned}$$

Recall that $c(u^k) := \log \rho(u^k)$. Note from (6.120) and estimate (5.10) that there exists $N > 0$ independent of k such that for each $s \in \mathbb{R}$, $k \in \mathbb{N}$,

$$\|v^k(s)\| \leq N(\mu(s))^{-\frac{1}{2}}. \qquad (6.122)$$

Hence there exist $s_1, s_2 \in \mathbb{R}$, $|s_i| \geq 1, i = 1, 2$, *independent* of $k \in \mathbb{N}$, such that

$$\begin{aligned} s \leq s_1 &\Rightarrow \|v^k(s)\| \leq \delta, \\ s \geq s_2 &\Rightarrow \|v^k(s)\| \leq \delta, \end{aligned} \qquad (6.123)$$

where $\delta > 0$ is as in Theorem 6.1.5. So for each $k \in \mathbb{N}, s \in \mathbb{R}$,

$$\|v^k(s)\| \leq \begin{cases} \kappa e^{-\alpha s_1} e^{\alpha s}, & s \leq s_1 \\ N\mu(s)^{-\frac{1}{2}}, & s_1 < s < s_2 \\ \kappa e^{\beta s_2} e^{-\beta s}, & s \geq s_2. \end{cases} \qquad (6.124)$$

Since the right-hand side of (6.124) defines a square-integrable function on $\mathbb{R}$, and $\|v^k(s) + \psi(s) - S\|$ is bounded independently of $s \in \mathbb{R}, k \in \mathbb{N}$, the form of ϕ given in (5.22) and Lebesgue's Dominated Convergence Theorem together yield from (6.121) that $\rho(u^k) \to 0$ if $s_k \to -\infty$.

Now for $i \in \{1, \ldots, N\}$, $\psi_i(s) = T_i$ when $s \leq -1$. Choose $\tilde{s}$ such that

$$s \leq \tilde{s} \Rightarrow \|v^k(s)\| \leq \frac{1}{2}|T_i - S_i|$$

for each $i \in \{1, \ldots, N\}$, $k \in \mathbb{N}$ (which is possible by (6.122). So there exists $\eta > 0$ such that for each $k \in \mathbb{N}$,

$$\|v^k(s) + \psi(s) - T\|^2 \geq \eta > 0 \text{ for each } s \leq \tilde{s}. \qquad (6.125)$$

Since $\phi(s) = 1$ for $s \geq 0$, there exists $t > 0$ such that

$$s_k \geq t \Rightarrow \phi(s + s_k) = 1 \text{ for } s \geq -t. \tag{6.126}$$

So if $s_k \to \infty$, For k sufficiently large, $\phi(s + s_k) = 1$ for $s \geq -s_k$. In particular, when $s_k \geq -\tilde{s}$,

$$\begin{aligned} \rho(u^k)^2 &\geq \int_{-s_k}^{-\tilde{s}} \|v^k(s) + \psi(s) - T\|^2 \phi(s + s_k)\, ds \\ &\geq \eta \int_{-s_k}^{-\tilde{s}} ds = \eta(s_k - \tilde{s}) \to \infty \text{ as } s_k \to \infty, \end{aligned} \tag{6.127}$$

since $\tilde{s}$ is independent of k.

It follows that $\rho(u^k) \to \infty$ as $s_k \to \infty$ and $\rho(u^k) \to 0$ as $s_k \to -\infty$. Hence since $|s_k| \to \infty$ as $k \to \infty$, $c(u^k)$ is unbounded. The result follows.

□

Chapter 7

Existence of monotone solutions

The first part of this chapter is devoted to proving that for each $R > 0$, there exists $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u'(s) + \psi'(s) > 0$ for each $s \in \mathbb{R}$, u satisfies (5.30), and $u(s) + \psi(s) \to S, T$ as $s \to -\infty, +\infty$. We then tie this existence result for the approximate system together with the uniform *a priori* estimates for monotone solutions of Chapter 6 to obtain existence of a monotone solution of the autonomous system (1.9).

7.1 Existence of monotone solutions of the approximate system

Fix $R > 0$ throughout this section. With the aim of constructing a set $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ satisfying (5.58), we show first that non-monotone solutions $w_N = u_N + \psi$ of (5.30) are bounded away from monotone solutions $w_M = u_M + \psi$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. In the following, write $u_M = w_M - \psi$ when $w'_M(s) > 0$ for each $s \in \mathbb{R}$ (that is, the subscript M means monotone), and $u_N = w_N - \psi$ when there exists $s_0 \in \mathbb{R}$ and $i \in \{1, \dots, N\}$ such that $(w_{N_i})'(s_0) \le 0$ (that is, the subscript N means non-monotone).

Theorem 7.1.1 *Let $R > 0$ be given. Then there exists $r > 0$ (dependent on R) such that if $u_M, u_N \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (5.31) with corresponding $\tau = \tau_M, \tau_N \in [0, 1]$ for this R, and $w_M = u_M + \psi$ and $w_N = u_N + \psi$ are monotone* *and non-monotone respectively, then*

$$\|u_M - u_N\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \geq r. \tag{7.1}$$

Proof. Suppose to the contrary, that there exist sequences $\{u^k_M\}_{k=1}^\infty, \{u^k_N\}_{k=1}^\infty \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N)$ with corresponding sequences $\{\tau_k\}, \{t_k\} \subset [0,1]$ such that u^k_M, u^k_N satisfy (5.31) with $\tau = \tau_k, t_k$ respectively and R as given, $w^k_M = u^k_M + \psi$ and $w^k_N = u^k_N + \psi$ are monotone and non-monotone respectively, and

$$\|u^k_M - u^k_N\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \to 0 \ \text{ as } \ k \to \infty. \tag{7.2}$$

By Theorem 6.2.3, there exists $C > 0$ such that $\|u^k_M\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} < C$ for each $k \in \mathbb{N}$. Thus taking a subsequence if necessary, it can be supposed that there exists $u^0_M \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ such that u^k_M converges weakly to u^0_M in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, and that $\tau_k \to \tau_0 \in [0,1]$ as $k \to \infty$. Theorem 5.2.7 yields the existence of a bounded linear positive-definite self-adjoint operator $S_\mu : W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ such that

$$(P_R(\tau_k, u^k_M))(S_\mu(u^k_M - u^0_M)) \geq \|u^k_M - u^0_M\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta(u^k_M, u^0_M, \tau_k) \tag{7.3}$$

where $\theta(u^k_M, u^0_M, \tau_k) \to 0$ as $k \to \infty$ (since $\theta(u, u_0, \tau) \to 0$ *uniformly* for $\tau \in [0,1]$ as $u \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$). Here $P_R : [0,1] \times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$ is as defined in (5.32), with R as fixed. Now because u^k_M solves (5.31) with $\tau = \tau_k$, Lemma 5.2.6 gives that $P_R(\tau_k, u^k_M) = 0$. So (7.3) yields that

$$\|u^k_M - u^0_M\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \to 0 \ \text{ as } \ k \to \infty. \tag{7.4}$$

Moreover, P_R is jointly continuous in τ and u by Lemma 5.2.2, and hence

$$P_R(\tau_0, u^0_M) = 0. \tag{7.5}$$

So $u_M^0 \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N)$ solves (5.31) with $\tau = \tau_0$, by Lemma 5.2.6. Put $w^0 := u_M^0 + \psi$, $w_M^k = u_M^k + \psi$, $w_N^k = u_N^k + \psi$. Then by (7.4) and (7.2),

$$\|w_M^k - w^0\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)},\ \ \|w_N^k - w^0\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \to 0 \ \text{ as } \ k \to \infty. \tag{7.6}$$

Estimate (5.10) thus gives that

$$\sup_{s\in\mathbb{R}} \|w_M^k(s) - w^0(s)\|, \sup_{s\in\mathbb{R}} \|w_N^k(s) - w^0(s)\| \to 0 \ \text{ as } \ k \to \infty. \tag{7.7}$$

Since $\|w^0(s)\|$ is bounded independently of s (by estimate (5.10), for example), it follows from (7.7) that $\|w_M^k(s)\|, \|w_N^k(s)\|$ are bounded independently of $s \in \mathbb{R}$ and $k \in \mathbb{N}$. Thus by Theorem 6.1.2, $\|\left(w_M^k\right)''(s)\|$, $\|\left(w_N^k\right)''(s)\|$, $\|(w_0)''(s)\|$ are bounded independently of $s \in \mathbb{R}$ and $k \in \mathbb{N}$. Landau's inequality then yields that

$$\sup_{s\in\mathbb{R}} \|\left(w_M^k - w^0\right)'(s)\|, \sup_{s\in\mathbb{R}} \|\left(w_N^k - w^0\right)'(s)\| \to 0 \ \text{ as } \ k \to \infty. \tag{7.8}$$

From (7.8) and the monotonicity of w_M^k, it follows that

$$\left(w^0\right)'(s) \geq 0 \ \text{ for all } \ s \in \mathbb{R}. \tag{7.9}$$

Since u_0 satisfies (5.31), Lemma 6.1.8 with $\sigma = \sigma_R$ thus yields that

$$\left(w^0\right)' > 0 \ \text{ for all } \ s \in \mathbb{R}, \tag{7.10}$$

because of the fact that $w^0 = u^0 + \psi$ where $u^0 \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ means that $w^0 \to S$ as $s \to -\infty$ and $w^0 \to T$ as $s \to \infty$, so w_0 is not identically constant.

We will show that (7.10) contradicts (7.8). Since $w_N^k(s) \to S$ as $s \to -\infty$, $w_N^k(s) \to T$ as $s \to \infty$, and w_N^k is not strictly monotone, there exists $s_k \in \mathbb{R}$ such that $\left(w_N^k\right)'(s_k)$ has at least one zero component. Taking a subsequence if necessary, it may be assumed without loss of generality that $\left(w_N^k\right)_1'(s_k) = 0$ for each k, and that $t_k \to t_0 \in [0,1]$ as $k \to \infty$.

Suppose first that the sequence $\{s_k\}$ is bounded. Then there is a subsequence such that $s_j \to s_0 \in \mathbb{R}$ as $j \to \infty$. By (7.8), it follows that $(w^0)_1'(s_0) = 0$. But this contradicts (7.10), so $\{s_k\}$ cannot be bounded.

Consider now the case when there is a subsequence of the s_k such that $s_k \to -\infty$ as $k \to \infty$; the case when $s_k \to \infty$ is similar, involving analysis at T instead of at S. Since f satisfies **(f1)** and **(f3)**, there exist $\lambda < 0$ and $q > 0$, $q \in \mathbb{R}^N$, such that

$$df[S]q = \lambda q, \tag{7.11}$$

by the Perron-Frobenius Theorem. So by virtue of the fact that f is continuously differentiable, there exists $\epsilon > 0$ such that

$$u_0 \in \mathbb{R}^N, \|u_0 - S\| < \epsilon \Rightarrow df[u_0]q < 0 \text{ and } df[u_0] \in P^{N\times N}. \tag{7.12}$$

Choose $s_* < 0$ sufficiently negative that

$$\begin{aligned} &(i) && s_* < -R-1 \text{ (so that } s_* \text{ lies outside the support of } \sigma_R\text{), and} && (7.13)\\ &(ii) && \|w^0(s) - S\| < \frac{\epsilon}{2} \text{ for } s \le s_* \text{ (which is possible since} && \\ & && w^0(s) = u_M^0(s) + \psi(s) \to S \text{ as } s \to -\infty). && (7.14) \end{aligned}$$

Also, using (7.14), (7.7) and (7.8) and the facts that $(w^0)'(s_*) > 0$ and $s_k \to -\infty$ as $k \to \infty$, we can choose $k_* \in \mathbb{N}$ sufficiently large that

$$\begin{aligned} &(i) && s_{k_*} < s_*, && (7.15)\\ &(ii) && \left(w_N^{k_*}\right)'(s_*) > 0, \text{ and} && (7.16)\\ &(iii) && \|w_N^{k_*}(s) - S\| < \epsilon \text{ for } s < s_*. && (7.17) \end{aligned}$$

Now consider the boundary value problem

$$\left.\begin{aligned} &Av''(s) + c\left(u_N^{k_*}\right)v'(s) + df[w_N^{k_*}(s)]v(s) = 0, \quad s \le s_*,\\ &v(s_*) = \left(w_N^{k_*}\right)'(s_*), \quad v(s) \to 0 \text{ as } s \to -\infty,\\ &v \in C^2((-\infty, s_*], \mathbb{R}^N). \end{aligned}\right\} \tag{7.18}$$

Differentiating (5.31) and taking (7.13) into account, it is clear that

$$v(s) = \left(w_N^{k_*}\right)'(s), \quad s \le s_*, \tag{7.19}$$

is a solution of (7.18). We will show that any solution v of (7.18) is *strictly* positive on $(-\infty, s_*]$, which contradicts (7.14) and the choice of the s_k.

So suppose that a solution v of (7.18) is not strictly positive for each $s \le s_*$. For $x \in \mathbb{R}$, define the vector-valued function v^x by

$$v^x(s) = v(s) + xq \tag{7.20}$$

where $q > 0, q \in \mathbb{R}^N$ is as in (7.11). Now for each $i \in \{1, \dots, N\}$, $v_i(s_*) > 0$ and $v_i(s) \to 0$ as $s \to -\infty$. So if for some $i \in \{1, \dots, N\}$, $s < s_*$, $v_i(s) \le 0$, then $\inf_{s \le s_*} v_i(s)$ is *attained* at some $t \in \mathbb{R}$, $-\infty < t < s_*$. Hence the hypothesis that v is not strictly positive for $s \le s_*$ implies the existence of $x_0 \in \mathbb{R}$, $x_0 \ge 0$, such that

$$v^{x_0}(s) \ge 0 \;\text{ for }\; -\infty \le s \le s_*, \tag{7.21}$$

and for some $\tilde{s}$, $-\infty < \tilde{s} < s_*$, $v^{x_0}(\tilde{s})$ has at least one zero component. (That $\tilde{s} < s_*$ follows from the fact that $v(s_*) = \left(w_N^{k_*}\right)'(s_*) > 0$.) Without loss of generality, say $v_1^{x_0}(\tilde{s}) = 0$. Whence

$$v_1(s) \ge -x_0 q_1 \;\text{ for }\; s \le s_* \;\text{ and }\; v_1(\tilde{s}) = -x_0 q_1, \tag{7.22}$$

so

$$v_1'(\tilde{s}) = 0 \;\text{ and }\; v_1''(\tilde{s}) \ge 0. \tag{7.23}$$

First consider the case when $x_0 > 0$. We aim to prove that

$$\sum_{i=1}^{N} \frac{\partial f_1}{\partial w_i}(w_N^{k_*}(\tilde{s})) v_i(\tilde{s}) > 0, \tag{7.24}$$

which contradicts the first equation of (7.18) since (7.23) holds. For this, consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(t) = \sum_{i=1}^{N} \frac{\partial f_1(w_N^{k_*}(\tilde{s}))}{\partial w_i}((1-t)v_i(\tilde{s}) - t x_0 q_i), \;\; t \in [0,1]. \tag{7.25}$$

If $v(\tilde{s}) = -x_0 q$, then (7.24) is a consequence of (7.11). Otherwise, (7.22) and the local monotonicity of f give that $\varphi'(t) \le 0$ for each $t \in (0,1)$, and (7.11) that $\varphi(1) > 0$. Whence $\varphi(0) > 0$; that is, (7.24) holds.

Suppose now that $x_0 = 0$, so that $v(s) \ge 0$ for $s \le s_*$ and $v_1(\tilde{s}) = 0$, for some $\tilde{s} < s_*$. If $v(\tilde{s}) \ne 0$, (7.24) follows from the local monotonicity of f, which contradicts (7.18) as above. Otherwise, $v(\tilde{s}) = 0$, so $v'(\tilde{s}) = 0$ (since $v(s) \geq 0$ for $s \leq s_*$). The uniqueness of the solution to the initial value problem

$$\left.\begin{array}{l} Av''(s) + c\left(u_N^{k_*}\right) v'(s) + df[w_N^{k_*}(s)]v(s) = 0, \\ v(\tilde{s}) = 0, \;\; v'(\tilde{s}) = 0 \end{array}\right\} \tag{7.26}$$

yields that $v(s) \equiv 0$ for $s \leq s_*$. This contradicts the boundary condition $v(s_*) = \left(w_N^{k_*}\right)'(s_*) > 0$. The theorem follows.

□

Now define $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ by

$$\Omega := \cup_{u \in \mathcal{M}_R} \{v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) : \|u - v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} < \frac{r}{2}\} \tag{7.27}$$

where $\mathcal{M}_R$ is as defined in (6.1), and $r > 0$ is as yielded by Theorem 7.1.1. Clearly Ω is open, and bounded by Theorem 6.2.3. Moreover, with $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ as constructed in Theorem 5.2.7, and P_R as defined by (5.32) for fixed $R > 0$,

$$0 \notin S^*_\mu P_R([0, 1] \times \partial\Omega) \tag{7.28}$$

by (5.58); if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies (5.31) for some $\tau \in [0, 1]$ and $w = u + \psi$ is non-monotone, then Theorem 7.1.1 yields that $u \notin \overline{\Omega}$; on the other hand, if $w = u + \psi$ is monotone, then $u \in \Omega^o$ (the interior of Ω). Hence (5.59) holds; that is,

$$deg_{(S)+}(S^*_\mu P_R(1, \cdot), \Omega, 0) = deg_{(S)+}(S^*_\mu P_R(0, \cdot), \Omega, 0). \tag{7.29}$$

Now when $\tau = 0$, (5.31) reduces to the system studied in [42]. We will use [42] to infer that $deg_{(S)+}(S^*_\mu P_R(1, \cdot), \Omega, 0) \neq 0$. Observe first that the Vol'perts' argument for existence in [42] can be simplified slightly by constructing $\tilde{\Omega}$ in a way analogous to the construction of Ω above. Then since there are no zeros of the homotopy function on $\partial\tilde{\Omega}$ for any $\tau \in [0, 1]$, it follows from homotopy invariance, and the analysis regarding the other endpoint of the homotopy, that

$$deg_{(S)+}(S^*_\mu P_R(0, \cdot), \tilde{\Omega}, 0) = 1. \tag{7.30}$$

(Note that the operator S_μ is constructed in this thesis in the same way as the construction in [42], correcting [42] as necessary; moreover, it is shown in [42] that the degree does not depend on the arbitrariness of the operator satisfying the conditions of Theorem 5.2.7.) Next, note that since all solutions u of $P_R(0,u) = 0$ with $w = u + \psi$ monotone are contained in $\Omega \cap \tilde{\Omega}$, and there are no non-monotone solutions of $P_R(0,u) = 0$ in $\overline{\Omega} \cup \overline{\tilde{\Omega}}$, the excision property of degree yields that

$$deg_{(S)_+}(S_\mu^* P_R(0,\cdot), \Omega, 0) = deg(S_\mu^* P_R(0,\cdot), \Omega \cap \tilde{\Omega}, 0) = deg_{(S)_+}(S_\mu^* P_R(0,\cdot), \tilde{\Omega}, 0). \tag{7.31}$$

Thus (7.29) and (7.30) imply that

$$deg_{(S)_+}(S_\mu^* P_R(1,\cdot), \Omega, 0) = 1. \tag{7.32}$$

The following is then immediate from the existence property of degree, (5.58), and the fact that there are no non-monotone solutions of (5.31) in $\overline{\Omega}$.

Theorem 7.1.2 *Let $R > 0$ be given. Then there exists $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ such that*

$$u'(s) + \psi'(s) > 0 \ \textit{for each} \ s \in \mathbb{R},$$

and

$$A(u'' + \psi'') + c(u)(u' + \psi') + \sigma_R G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0.$$

7.2 Existence of monotone solutions of the autonomous system

Here we combine the uniform *a priori* estimates for monotone solutions of Chapter 6 with the existence of monotone solutions for the approximate system proved in section 7.1. The goal of Chapters 5-7 is the following.

Theorem 7.2.1 *Let $A \in M^{N\times N}$ be a positive-definite diagonal matrix, $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfy* **(f1)**-**(f4)**, *and $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ satisfy* **(G1)**-**(G3)**. *Then there exists $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ and $c \in \mathbb{R}$ such that*

$$u'(s) + \psi'(s) > 0 \ \textit{ for each } \ s \in \mathbb{R}, \tag{7.33}$$

and

$$A(u'' + \psi'') + c[u' + \psi'] + G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0. \tag{7.34}$$

Proof. For each $n \in \mathbb{N}$, let $u_n \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ be such that

$$u_n'(s) + \psi'(s) > 0 \ \text{ for each } \ s \in \mathbb{R}, \tag{7.35}$$

and

$$A(u_n''+\psi'')+c(u_n)[u_n'+\psi']+\sigma_n G(u_n+\psi, u_n'+\psi')(u_n'+\psi')+f(u_n+\psi) = 0. \tag{7.36}$$

Such solutions exist by Theorem 7.1.2. By Theorem 6.1.5, we can take a subsequence, if necessary, so that there exists $c_0 \in \mathbb{R}$ with

$$c(u_n) \to c_0 \ \text{ as } \ n \to \infty. \tag{7.37}$$

Also, $\{u_n\}_{n=1}^{\infty}$ is bounded in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ *independently* of $n \in \mathbb{N}$ by Theorem 6.2.3. So since closed balls in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ are weakly sequentially compact, there exists $u_0 \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ such that

$$u_n \rightharpoonup u_0 \ \text{ in } \ W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \ \text{ as } \ n \to \infty. \tag{7.38}$$

Now let $r > 0$ and let $Y_r := [-r, r]$. Note that $u_n \rightharpoonup u_0$ in $W^1_2(\mathbb{R}, \mathbb{R}^N)$ as $n \to \infty$. So the Sobolev Embedding Theorem yields that

$$u_n \to u_0 \ \text{ in } \ C(Y_r, \mathbb{R}^N) \ \text{ as } \ n \to \infty. \tag{7.39}$$

Since $w_n = u_n + \psi$ is monotone for each n, $\|u_n\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}$ is bounded independently of n, and thus, by Theorem 6.1.2, so are $\|u_n'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}$ and $\|u_n''\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}$. The Arzela-Ascoli theorem can thus be applied to yield that there is a subsequence $\{u_k\}$ of $\{u_n\}$ such that u_k converges in $C^1(Y_r, \mathbb{R}^N)$. Now recall from the construction of σ_R in section 5.2.1 that for each $s \in \mathbb{R}$, $\sigma_n \to 1$ as $n \to \infty$. It thus follows from (7.36), (7.37), the continuity of f and G and the convergence of u_k that u_k'' is uniformly convergent on Y_r. Hence there is a subsequnce $\{u_l\}$ of $\{u_k\}$ and $\tilde{u} \in C^2(Y_r, \mathbb{R}^N)$, such that

$$u_l \to \tilde{u} \ \text{ in } \ C^2(Y_r, \mathbb{R}^N) \ \text{ as } \ l \to \infty. \tag{7.40}$$

A diagonal sequence argument yields the existence of a subsequence $\{u_j\}$ of $\{u_l\}$ that converges in $C^2(Y_r, \mathbb{R}^N)$ for each $r > 0$ to a limit $u \in C^2(\mathbb{R}, \mathbb{R}^N)$. But since $u_n \to u_0$ in $C(Y_r, \mathbb{R}^N)$ as $n \to \infty$ for each $r > 0$ (7.39), uniqueness of limits in $C(Y_r, \mathbb{R}^N)$ yields that

$$u = u_0. \tag{7.41}$$

For each $s \in \mathbb{R}$, $\sigma_n(s) \to 1$ as $n \to \infty$. Thus for each $s \in \mathbb{R}$, we can pass to the limit in (7.36)as $j \to \infty$ along the subsequence for which $u_j \to u_0$ to obtain that

$$A(u_0'' + \psi'') + c_0[u_0' + \psi'] + G(u_0 + \psi, u_0' + \psi')(u_0' + \psi') + f(u_0 + \psi) = 0. \tag{7.42}$$

Since $u_0 \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, it follows immediately from estimate (5.10) that $|u_0(s)| \to 0$ as $|s| \to \infty$, and hence that $w_0(s) := u_0(s) + \psi(s) \to S$ as $s \to -\infty$ and $w_0(s) \to T$ as $s \to \infty$, by (5.22). Also, since $u_j'(s) \to u_0'(s)$ as $j \to \infty$ for each $s \in \mathbb{R}$, and $u_j'(s) + \psi'(s) > 0$ for each j, s, we have that $w_0'(s) \geq 0$ for each s. Applying Lemma 6.1.8 with $\sigma \equiv 1$ thus yields that $w_0'(s) > 0$ for each $s \in \mathbb{R}$, since $w_0(s)$ is not identically constant. The result follows.

□

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